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Day and Thomson's Series.

A

TREATISE

OF

PLANE TRIGONOMETRY,

AND THE

MENSURATION OF HEIGHTS AND DISTANCES.

TO WHICH IS PREFIXED

A SUMMARY VIEW OF THE NATURE AND USE

OF

LOGARITHMS.

ADAPTED TO

THE METHOD OF INSTRUCTION IN SCHOOLS AND ACADEMIES

BY JEREMIAH DAY, D.D. LL.D.
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In Memoriam
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PLANE TRIGONOMETRY.

SCARCELY any department of Mathematics is more important, or more extensive in its applications, than Trigonometry. By it the mariner traces his path on the ocean; the geographer determines the latitude and longitude of places, the dimensions and positions of countries, the altitude of mountains, the courses of rivers, &c., and the astronomer calculates the distances and magnitudes of the heavenly bodies, predicts the eclipses of the sun and moon, and measures the progress of light from the stars.

The section on right angled triangles in this treatise, may perhaps be considered as needlessly minute. The solutions might, in all cases, be effected by the theorems which are given for oblique angled triangles. But the applications of rectangular trigonometry are so numerous, in navigation, surveying, astronomy, &c., that it was deemed important, to render familiar the various methods of stating the relations of the sides and angles; and especially to bring distinctly into view the principle on which most trigonometrical calculations are founded, the proportion between the parts of the given triangle, and a similar one formed from the sines, tangents, &c., in the tables.

As this treatise is intended to form a part of Day and Thomson's Course of Mathematics for the use of Schools and Academies, the references to Algebra are made to Thomson's Abridgment; and the references to Geometry, to Thomson's Legendre, as well as to Euclid's Elements.

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LOGARITHMS.

SECTION I.

NATURE OF LOGARITHMS.

ART. 1. The operations of Multiplication and Division, when they are to be often repeated, become so laborious, that it is an object of importance to substitute, in their stead, more simple methods of calculation, such as Addition and Subtraction. If these can be made to perform, in an expeditious manner, the office of multiplication and division, a great portion of the time and labor which the latter processes require, may be saved.

Now it has been shown, (Algebra, 189, 193,) that *powers* may be multiplied by adding their *exponents,* and divided, by subtracting their exponents. In the same manner, *roots* may be multiplied and divided, by adding and subtracting their fractional exponents. (Alg., 232, 239.) When these exponents are arranged in tables, and applied to the general purposes of calculation, they are called *Logarithms.*

2. LOGARITHMS, THEN, ARE THE EXPONENTS OF A SERIES OF POWERS AND ROOTS.

In forming a system of logarithms, some particular number is fixed upon, as the *base, radix,* or first power, whose logarithm is always 1. From this a series of powers is raised, and the exponents of these are arranged in tables for use. To explain this, let the number which is chosen for the first power be represented by a. Then taking a series of powers, both direct and reciprocal, as in Alg. 163;

$$a^4,\ a^3,\ a^2,\ a^1,\ a^0,\ a^{-1},\ a^{-2},\ a^{-3},\ a^{-4},\ \&c.$$

The logarithm of a^3 is 3, and the logarithm of a^{-1} is—1,

of a^1 is 1, of a^{-2} is—2,
of a^0 is 0, of a^{-3} is—3, &c.

Universally, the logarithm of a^x is x.

3. In the system of logarithms in common use, called *Briggs's* logarithms, the number which is taken for the radix or base is 10. The above series, then, by substituting 10 for a, becomes

$$10^4,\ 10^3,\ 10^2,\ 10^1,\ 10^0,\ 10^{-1},\ 10^{-2},\ 10^{-3},\ \&c.$$

$$\text{Or } 10000,\ 1000,\ 100,\ 10,\ 1,\ \tfrac{1}{10},\ \tfrac{1}{100},\ \tfrac{1}{1000},\ \&c.$$

Whose logarithms are

4, 3, 2, 1, 0, —1, —2, —3, &c.

4. The fractional exponents of *roots*, and of powers of roots, are converted into *decimals*, before they are inserted in the logarithmic tables. See Alg. 208.

The logarithm of $a^{\frac{1}{3}}$, or $a^{0.3333}$, is 0.3333,

of $a^{\frac{2}{3}}$, or $a^{0.6666}$, is 0.6666,

of $a^{\frac{3}{7}}$, or $a^{0.4285}$, is 0.4285,

of $a^{\frac{11}{3}}$, or $a^{3.6666}$, is 3.6666, &c.

These decimals are carried to a greater or less number of places, according to the degree of accuracy required.

5. In forming a system of logarithms, it is necessary to obtain the logarithm of each of the numbers in the natural series 1, 2, 3, 4, 5, &c.; so that the logarithm of any number may be found in the tables. For this purpose, the *radix* of the system must first be determined upon; and then every other number may be considered as some power or root of

this. If the radix is 10, as in the common system, every other number is to be considered as some power of 10.

If the exponent is a fraction, and the *numerator* be increased, the power will be increased; but if the *denominator* be increased, the power will be diminished.

6. To obtain then the logarithm of any number, according to Briggs's system, we have to find a power or root of 10 which shall be equal to the proposed number. The *exponent* of that power or root is the logarithm required. Thus

$$\left.\begin{array}{r}7=10^{0.8451}\\20=10^{1.3010}\\30=10^{1.4771}\\400=10^{2.6020}\end{array}\right\}\begin{array}{c}\text{therefore the}\\\text{logarithm}\end{array}\left\{\begin{array}{l}\text{of } 7 \text{ is } 0.8451\\\text{of } 20 \text{ is } 1.3010\\\text{of } 30 \text{ is } 1.4771\\\text{of } 400 \text{ is } 2.6020\text{, \&c.}\end{array}\right.$$

7. A logarithm generally consists of two parts, an *integer* and a *decimal*. Thus the logarithm 2.60206, or, as it is sometimes written, 2+.60206, consists of the integer 2, and the decimal .60206. The integral part is called the *characteristic* or *index*[*] of the logarithm; and is frequently omitted, in the common tables, because it can be easily supplied, whenever the logarithm is to be used in calculation.

By art. 3d, the logarithms of

	10000,	1000,	100,	10,	1,	.1,	.01,	.001, &c.
are	4,	3,	2,	1,	0,	—1,	—2,	—3, &c.

As the logarithms of 1 and of 10 are 0 and 1, it is evident, that, if any given number be between 1 and 10, its logarithm will be between 0 and 1, that is, it will be greater than 0, but less than 1. It will therefore have 0 for its index, with a decimal annexed.

Thus, the logarithm of 5 is 0.69897.

---

[*] The term *index*, as it is used here, may possibly lead to some confusion in the mind of the learner. For the logarithm itself is the index or exponent of a power. The characteristic, therefore, is the index of an index.

For the same reason, if the given number be between

$$\left.\begin{array}{r}10\text{ and }100,\\100\text{ and }1000,\\1000\text{ and }10000,\end{array}\right\}\begin{array}{c}\text{the log.}\\\text{will be}\\\text{between}\end{array}\left\{\begin{array}{ll}1\text{ and }2,\text{ i.e.} & 1+\text{the dec. part.}\\2\text{ and }3, & 2+\text{the dec. part.}\\3\text{ and }4, & 3+\text{the dec. part.}\end{array}\right.$$

We have, therefore, when the logarithm of an integer or mixed number is to be found, this general rule:

8. *The index of the logarithm is always one less, than the number of integral figures, in the natural number whose logarithm is sought:* or, the index shows how far the first figure of the natural number is removed from the place of units.

Thus, the logarithm of 37 is 1.56820.

Here, the number of figures being *two*, the index of the logarithm is 1.

The logarithm of 253 is 2.40312.

Here the proposed number 253 consists of *three* figures, the first of which is in the second place from the unit figure. The index of the logarithm is therefore 2.

The logarithm of 62.8 is 1.79796.

Here it is evident that the mixed number 62.8 is between 10 and 100. The index of its logarithm must, therefore, be 1.

9. As the logarithm of 1 is 0, the logarithm of a number *less* than 1, that is, of any proper *fraction*, must be *negative*.

Thus, by art. 3d,

The logarithm of $\frac{1}{10}$ or .1 is -1,
of $\frac{1}{100}$ or .01 is -2,
of $\frac{1}{1000}$ or .001 is -3, &c.

10. If the proposed number is *between* $\frac{1}{100}$ and $\frac{1}{1000}$, its logarithm must be between -2 and -3. To obtain the logarithm, therefore, we must either *subtract* a certain fractional part from -2, or *add* a fractional part to -3; that

is, we must either annex a *negative decimal* to —2, or a *positive one* to —3.

Thus, the logarithm
of .008 is either —2 —.09691, or —3+90309.*

The latter is generally most convenient in practice, and is more commonly written $\overline{3}.90309$. The line over the index denotes, that *that* is negative, while the *decimal* part of the logarithm is positive.

$$\text{The logarithm} \left\{ \begin{array}{l} \text{of } 0.3, \quad \text{is } \overline{1}.47712, \\ \text{of } 0.06, \ \ \text{is } \overline{2}.77815, \\ \text{of } 0.009, \text{is } \overline{3}.95424, \end{array} \right.$$

And universally,

11. *The negative index of a logarithm shows how far the first significant figure of the natural number, is removed from the place of units, on the right;* in the same manner as a *positive* index shows how far the first figure of the natural number is removed from the place of units on the left. (Art. 8.) Thus, in the examples in the last article,

The decimal 3 is in the *first* place from that of units,
6 is in the *second* place,
9 is in the *third* place;

And the indices of the logarithms are $\overline{1}$, $\overline{2}$, and $\overline{3}$.

12. It is often more convenient, however to make the *index* of the logarithm positive, as well as the decimal part. This is done by adding 10 to the index.

Thus, for —1, 9 is written, for —2, 8, &c.
Because —1+10=9, —2+10=8, &c.

---

* That these two expressions are of the same value will be evident, if we subtract the same quantity, +.90309 from each. The remainders will be equal, and therefore the quantities from which the subtraction is made must be equal.

With this alteration,

$$\text{The logarithm} \left\{ \begin{array}{l} \bar{1}.90309 \\ \bar{2}.90309 \\ \bar{3}.90309 \end{array} \right\} \text{becomes} \left\{ \begin{array}{l} 9.90309, \\ 8.90309, \\ 7.90309, \text{\&c.} \end{array} \right.$$

This is making the index of the logarithm 10 too great. But with proper caution, it will lead to no error in practice.

13. The *sum* of the logarithms of two numbers, is the logarithm of the *product* of those numbers; and the *difference* of the logarithms of two numbers, is the logarithm of the *quotient* of one of the numbers divided by the other. (Art. 2.) In Briggs's system, the logarithm of 10 is 1. (Art. 3.) If therefore any number be multiplied or divided by 10, its logarithm will be increased or diminished by 1: and as this is an integer, it will only change the *index* of the logarithm, without affecting the decimal part.

Thus, the logarithm of 4730 is 3.67486
And the logarithm of 10 is 1.
The logarithm of the product 47300 is 4.67486
And the logarithm of the quotient 473 is 2.67486

Here the *index* only is altered, while the decimal part remains the same. We have then this important property,

14. The DECIMAL PART *of the logarithm of any number is the same, as that of the number multiplied or divided by* 10, 100, 1000, &c.

Thus the log. of	is	
45670,	4.65963,	
4567,	3.65963,	
456.7,	2.65963,	
45.67,	1.65963,	
4.567,	0.65963,	
.4567,	$\bar{1}$.65963, or	9.65963,
.04567,	$\bar{2}$.65963,	8.65963,
.004567,	$\bar{3}$.65963,	7.65963

This property, which is peculiar to Briggs's system, is of great use in abridging the logarithmic tables. For when we have the logarithm of any number, we have only to change the index, to obtain the logarithm of every other number, whether integral, fractional, or mixed, consisting of the same significant figures. The decimal part of the logarithm of a fraction found in this way, is always *positive.* For it is the same as the decimal part of the logarithm of a whole number.

17. *If a series of numbers be in* GEOMETRICAL *progression, their logarithms will be in* ARITHMETICAL *progression.* For, in a geometrical series ascending, the quantities increase by a common *multiplier ;* (Alg. 359.) That is, each succeeding term is the *product* of the preceding term into the ratio. But the *logarithm* of this product is the *sum* of the logarithms of the preceding term and the ratio ; that is, the logarithms increase by a common *addition,* and are, therefore, in arithmetical progression. (Alg. 326.) In a geometrical progression *descending,* the terms decrease by a common *divisor,* and their logarithms, by a common *difference.**

Thus, the numbers 1, 10, 100, 1000, 10000, &c., are in geometrical progression.

And their logarithms 0, 1, 2, 3, 4, &c., are in arithmetical progression.

---

* See Note A.

SECTION II.

DIRECTIONS FOR TAKING LOGARITHMS AND THEIR NUMBERS FROM THE TABLES.*

Art. 24. The purpose which logarithms are intended to answer, is to enable us to perform arithmetical operations with *greater expedition,* than by the common methods. Before any one can avail himself of this advantage, he must become so familiar with the tables, that he can readily find the logarithm of any number; and, on the other hand, the number to which any logarithm belongs.

In the common tables, the *indices* to the logarithms of the first 100 numbers are inserted. But, for all other numbers, the *decimal* part only of the logarithm is given; while the index is left to be supplied, according to the principles in Arts. 8 and 11.

25. *To find the logarithm of any number between* 1 *and* 100:

Look for the proposed number, on the left; and against it, in the next column, will be the logarithm, with its index. Thus,

The log. of 18 is 1.25527. The log. of 73 is 1.86332.

26. *To find the logarithm of any number between* 100 *and* 1000; or of any number consisting of not more than three significant figures, with ciphers annexed.

---

* The best English Tables are Hutton's in 8vo. and Taylor's in 4to. In these, the logarithms are carried to seven places of decimals, and proportional parts are placed in the margin. The smaller tables are numerous; and, when accurately printed, are sufficient for common calculations.

The best log. tables for Surveyors are
n Gummere's Surveying – They are the
est because they contain the Secants
d Cosecants –

In the smaller tables, the three first figures of each number, are generally placed in the left hand column; and the fourth figure is placed at the head of the other columns.

Any number, therefore, between 100 and 1000, may be found on the left hand; and directly opposite, in the next column, is the decimal part of its logarithm. To this the *index* must be prefixed, according to the rule in Art. 8.

The log. of 458 is 2.66087, The log. of 935 is 2.97081,
of 796 2.90091, of 386 2.58659.

If there are *ciphers* annexed to the significant figures, the logarithm may be found in a similar manner. For, by Art. 14, the *decimal* part of the logarithm of any number is the same, as that of the number multiplied into 10, 100, &c. All the difference will be in the *index;* and this may be supplied by the same general rule.

The log. of 4580 is 3.66087, The log. of 326000 is 5.51322,
of 79600 4.90091, of 8010000 6.90363.

27. *To find the logarithm of any number consisting of* FOUR *figures, either with, or without, ciphers annexed.*

Look for the three first figures, on the left hand, and for the fourth figure, at the head of one of the columns. The logarithm will be found, opposite the three first figures, and in the column which, at the head, is marked with the fourth figure.*

The log. of 6234 is 3.79477, The log. of 783400 is 5.89398,
of 5231 3.71858, of 6281000 6.79803.

28. *To find the logarithm of a number containing* MORE *than* FOUR *significant figures.*

By turning to the tables, it will be seen, that if the *differences* between several numbers be small, in comparison with the numbers themselves; the differences of the *logarithms* will be nearly proportioned to the differences of the *numbers*. Thus,

---

* In Taylor's, Hutton's, and other tables, *four* figures are placed in the left hand column, and the *fifth* at the top of the page.

The log. of 1000 is 3.00000,
of 1001 3.00043,
of 1002 3.00087,
of 1003 3.00130,
of 1004 3.00173, &c.

Here the differences in the numbers are, 1, 2, 3, 4, &c., and the corresponding differences in the logarithms, are 43, 87, 130, 173, &c.

Now 43 is nearly half of 87, one-third of 130, one-fourth of 173, &c

Upon this principle, we may find the logarithm of a number which is between two other numbers whose logarithms are given by the tables. Thus, the logarithm of 21716 is not to be found in those tables which give the numbers to four places of figures only.

But by the table, the log. of 21720 is 4.33686
and the log. of 21710 is 4.33666

The difference of the two numbers is 10; and that of the logarithms 20.

Also, the difference between 21710, and the proposed number 21716, is 6.

If, then, a difference of 10 in the numbers
make a difference of 20 in the logarithms:
A difference of 6 in the numbers will
make a difference of 12 in the logarithms.

That is, 10 : 20 : : 6 : 12.

If, therefore, 12 be added to 4.33666, the log. of 21710;
The sum will be 4.33678, the log. of 21716.
We have, then, this

RULE.

To find the logarithm of a number consisting of more than four figures:

Take out the logarithm of two numbers, one greater, and the other less, than the number proposed: Find the difference of the two numbers, and the difference of their logarithms: Take also the difference between the least of the two numbers, and the proposed number. Then say,

As the difference of the two numbers,
To the difference of their logarithms;
So is the difference between the least of the two numbers, and the proposed number,
To the proportional part to be added to the least of the two logarithms.

It will generally be expedient to make the *first four figures*, in the least of the two numbers, the same as in the proposed number, substituting ciphers, for the remaining figures; and to make the greater number the same as the less, with the addition of a unit to the last significant figure. Thus,

For 36843, take 36840, and 36850,
For 792674, 792600, 792700,
For 6537825, 6537000, 6538000, &c.

The first term of the proportion will then be 10, or 100, or 1000, &c.

Ex. 1. Required the logarithm of 362572.

The logarithm of 362600 is 5.55943
of 362500 5.55931
The differences are 100, and 12.

Then 100 : 12 : : 72 : 8.64, or 9 nearly.
And the log. $5.55931+9=5.55940$, the log. required.

Ex. 2. The log. of 78264 is 4.89356
3. The log. of 143542 is 5.15698
4. The log. of 1129535 is 6.05290.

By a little practice, such a facility in abridging these calculations may be acquired, that the logarithms may be taken out, in a very short time. When great accuracy is not required, it will be easy to make an allowance sufficiently near, without formally stating a proportion. In the larger tables, the proportional parts which are to be added to the logarithms, are already prepared, and placed in the margin.

29. *To find the logarithm of a* DECIMAL FRACTION.

The logarithm of a decimal is the same as that of a whole number, excepting the *index*. (Art. 14.) To find then the logarithm of a decimal, take out that of a whole number consisting of the same figures; *observing to make the negative index equal to the distance of the first significant figure of the fraction from the place of units.* (Art. 11.)

The log. of 0.07643, is $\bar{2}.88326$, or 8.88326, (Art. 12.)
of 0.00259, $\bar{3}.41330$, or 7.41330,
of 0.0006278, $\bar{4}.79782$, or 6.79782.

30. *To find the logarithm of a* MIXED *decimal number.*

Find the logarithm, in the same manner as if *all* the figures were integers; and then prefix the index which belongs to the *integral* part, according to Art. 8.

The logarithm of 26.34 is 1.42062.

The index here is 1, because 1 is the index of the logarithm of every number greater than 10, and less than 100, (Art. 7.)

The log. of 2.36 is 0.37291, The log. of 364.2 is 2.56134,
of 27.8 1.44404, of 69.42 1.84148.

31. *To find the logarithm of a* VULGAR FRACTION.

From the nature of a vulgar fraction, the numerator may be considered as a *dividend*, and the denominator as a *divisor;* in other words, the value of the fraction is equal to the quotient of the numerator divided by the denominator. (Alg. 110.) But in logarithms, division is performed by *subtraction;* that is, the *difference* of the logarithms of two numbers, is the logarithm of the *quotient* of those numbers.

(Art. 1.) To find then the logarithm of a vulgar fraction, *subtract the logarithm of the denominator* from *that of the numerator.* The difference will be the logarithm of the fraction. Or the logarithm may be found, by first reducing the vulgar fraction to a *decimal.* If the numerator is less than the denominator, the index of the logarithm must be *negative,* because the value of the fraction is less than a unit. (Art. 9.)

Required the logarithm of $\frac{34}{87}$.

The log. of the numerator is 1.53148
of the denominator 1.93952

of the fraction 1.59196, or 9.59196.
The logarithm of $\frac{362}{7854}$ is $\bar{2}$.66362, or 8.66362.
of $\frac{7}{6329}$ $\bar{3}$.04376, or 7.04376.

32. If the logarithm of a *mixed number* is required, reduce it to an improper fraction, and then proceed as before.

The logarithm of $3\frac{7}{9}=\frac{34}{9}$ is 0.57724.

33. *To find the* NATURAL NUMBER *belonging to any logarithm.*

In computing by logarithms, it is necessary, in the first place, to take from the tables the logarithms of the numbers which enter into the calculation; and, on the other hand, at the close of the operation, to find the number belonging to the logarithm obtained in the result. This is evidently done by *reversing* the methods in the preceding articles.

Where great accuracy is not required, look in the tables for the logarithm which is *nearest* to the given one; and directly opposite on the left hand, will be found the *three first* figures, and at the top, over the logarithm, the *fourth* figure of the number required. This number, by pointing off decimals, or by adding ciphers, if necessary, must be made to correspond with the *index* of the given logarithm, according to Arts. 8 and 11.

The natural number belonging

to 3.86493 is 7327, to 1.62572 is 42.24,
to 2.90141 796.9, to $\bar{2}.89115$ 0.07783.

In the last example, the index requires that the first significant figure should be in the *second* place from units, and therefore a cipher must be prefixed. In other instances, it is necessary to annex ciphers on *the right*, so as to make the number of figures exceed the index by 1.

The natural number belonging

to 6.71567 is 5196000, to $\bar{3}.65677$ is 0.004537,
to 4.67062 46840, to $\bar{4}.59802$ 0.0003963.

34. When great accuracy is required, and the given logarithm is not exactly, or very nearly, found in the tables, it will be necessary to reverse the rule in Art. 28.

Take from the tables two logarithms, one the next greater, the other the next less than the given logarithm. Find the difference of the two logarithms, and the difference of their natural numbers; also the difference between the least of the two logarithms, and the given logarithm. Then say,

As the difference of the two logarithms,
To the difference of their numbers;
So is the difference between the given
logarithm and the least of the other two,
To the proportional part to be added to
the least of the two numbers.

Required the number belonging to the logarithm 2.67325.

Next great. log.	2.67330.	Its numb.	471.3.	Given log.	2.67325.
Next less	2.67321.	Its numb.	471.2.	Next less	2.67321.
Differences	9		0.1		4

Then, 9 : 0.1 : : 4 : 0.044, which is to be added
to the number 471.2

The number required is $\overline{471.244.}$

The natural number belonging
to 4.37627 is 23783.45, to 1.73698 is 54.57357,
to 3.69479 4952.08, to $\bar{1}$.09214 0.123635.

35. *Correction of the Tables.*—The tables of logarithms have been so carefully and so repeatedly calculated, by the ablest computers, that there is no room left to question their general correctness. They are not, however, exempt from the common imperfections of the press. But an error of this kind is easily corrected, by comparing the logarithm with any two others to whose *sum* or *difference* it ought to be equal. (Art. 1.)

Thus $48=24\times2=16\times3=12\times4=8\times6$. Therefore, the logarithm of 48 is equal to the *sum* of the logarithms of 24 and 2, of 16 and 3, &c.

And, $3=\frac{6}{2}=\frac{12}{4}=\frac{15}{5}=\frac{18}{6}=\frac{21}{7}$, &c. Therefore, the logarithm of 3 is equal to the *difference* of the logarithms of 6 and 2, of 12 and 4, &c.

SECTION III.

METHODS OF CALCULATING BY LOGARITHMS.

ART. 36. The arithmetical operations for which logarithms were originally contrived, and on which their great utility depends, are chiefly multiplication, division, involution, evolution, and finding the term required in single and compound proportion. The principle on which all these calculations are conducted, is this:

If the logarithms of two numbers be added, the SUM *will be the logarithm of the* PRODUCT *of the numbers;* and,

If the logarithm of one number be subtracted from that of another, the DIFFERENCE *will be the logarithm of the* QUOTIENT *of one of the numbers divided by the other.*

In proof of this, we have only to call to mind, that logarithms are the EXPONENTS *of a series of powers and roots.* (Arts. 2, 5.) And it has been shown, that powers and roots are *multiplied* by *adding* their exponents; and *divided*, by *subtracting* their exponents. (Alg. 189, 193, 232, 239.)

MULTIPLICATION BY LOGARITHMS.

37. ADD THE LOGARITHMS OF THE FACTORS: THE SUM WILL BE THE LOGARITHM OF THE PRODUCT.

In making the addition, 1 is to be carried for every 10, from the decimal part of the logarithm, to the index. (Art. 7.)

	Numbers.	Logarithms.		Numbers.	Logarithms.
Mult.	36.2 (Art. 30.)	1.55871	Mult.	640	2.80618
Into	7.84	0.89432	Into	2.316	0.36474
Prod.	283.8	2.45303	Prod.	1482	3.17092

The logarithms of the two factors are taken from the tables. The product is obtained, by finding, in the tables, the natural number belonging to the sum. (Art. 33.)

Mult.	89.24	1.95056	Mult.	134.	2.12710
Into	3.687	0.56667	Into	25.6	1.40824
Prod.	329.	2.51723	Prod.	3430	3.53534

38. When any or all of the indices of the logarithms are *negative*, they are to be added according to the rules for the addition of positive and negative quantities in algebra. But it must be kept in mind, that the decimal part of the logarithm is *positive*. (Art. 10.) Therefore, that which is carried from the decimal part to the index, must be considered positive also.

Mult.	62.84	1.79824	Mult.	0.0294	$\bar{2}.46835$
Into	0.682	$\bar{1}.83378$	Into	0.8372	$\bar{1}.92283$
Prod.	42.86	1.63202	Prod.	0.0246	$\bar{2}.39118$

In each of these examples, +1 is to be carried from the decimal part of the logarithm. This, added to —1, the lower index, makes it 0; so that there is nothing to be added to the upper index.

If any perplexity is occasioned, by the addition of positive and negative quantities, it may be avoided, by borrowing 10 to the index. (Art. 12.)

Mult.	62.84	1.79824	Mult.	0.0294	8.46835
Into	0.682	9.83378	Into	0.8372	9.92283
Prod.	42.86	1.63202	Prod.	0.0246	8.39118

Here 10 is added to the negative indices, and afterwards rejected from the index of the sum of the logarithms.

Multiply	26.83	1.42862		1.42862
Into	0.00069	$\bar{4}.83885$	or	6.83885
Product	0.01851	$\bar{2}.26747$		8.26747

Here +1 carried to —4 makes it —3, which added to the upper index +1, gives —2 for the index of the sum.

Multiply	.00845	$\bar{3}.92686$	or	7.92686
Into	1068.	3.02857		3.02857
Product	9.0246	0.95543		0.95543

The product of 0.0362 into 25.38 is 0.9188
of 0.00467 into 348.1 is 1.626
of 0.0861 into 0.00843 is 0.0007258

39. *Any number of factors* may be multiplied together, by adding their logarithms. If there are several *positive*, and several *negative* indices, these are to be reduced to one, as in algebra, by taking the difference between the sum of those which are negative, and the sum of those which are positive, increased by what is carried from the decimal part of the logarithms. (Alg. 53.)

Multiply	6832	3.83455		3.83455
Into	0.00863	$\bar{3}.93601$	or	7.93601
And	0.651	$\bar{1}.81358$		9.81358
And	0.0231	$\bar{2}.36361$	or	8.36361
And	62.87	1.79844		1.79844
Prod.	55.74	1.74619		1.74619

Ex. 2. The prod. of 36.4×7.82×68.91×0.3846 is 7544.

3. The prod. of 0.00629×2.647×0.082×278.8×0.00063 is 0.0002398.

40. *Negative* quantities are multiplied, by means of logarithms, in the same manner as those which are positive. (Art. 16.) But, after the operation is ended, the proper sign must be applied to the natural number expressing the product, according to the rules for the multiplication of positive and negative quantities in algebra. The negative index of a *logarithm*, must not be confounded with the sign which denotes that the *natural number* is negative. That which the index

of the logarithm is intended to show, is not whether the natural number is *positive or negative*, but whether it is *greater or less than a unit.* (Art. 16.)

Mult.	+36.42	1.56134	Mult.	—2.681	0.42830
Into	—67.31	1.82808	Into	+37.24	1.57101
Prod.	— 2451	3.38942	Prod.	—99.84	1.99931

In these examples, the logarithms are taken from the tables, and added, in the same manner, as if both factors were positive. But after the product is found, the negative sign is prefixed to it, because + is multiplied into —. (Alg. 82.)

Mult.	0.263	$\bar{1}.41996$	Mult.	0.065	$\bar{2}.81291$
Into	0.00894	$\bar{3}.95134$	Into	0.693	$\bar{1}.84073$
Prod.	0.002351	$\bar{3}.37130$	Prod.	0.04504	$\bar{2}.65364$

Here the indices of the logarithms are negative, but the product is positive, because the factors are both positive.

Mult.	—62.59	1.79650	Mult.	—68.3	1.83442
Into	—0.00863	$\bar{3}.93601$	Into	—0.0096	$\bar{3}.98227$
Prod.	+0.5402	$\bar{1}.73251$	Prod.	+0.6557	$\bar{1}.81669$

DIVISION BY LOGARITHMS.

41. FROM THE LOGARITHM OF THE DIVIDEND, SUBTRACT THE LOGARITHM OF THE DIVISOR; THE DIFFERENCE WILL BE THE LOGARITHM OF THE QUOTIENT. (Art. 36.)

	Numbers.	Logarithms.		Numbers.	Logarithms.
Divide	6238	3.79505	Divide	896.3	2.95245
By	2982	3.47451	By	9.847	0.99330
Quot.	2.092	0.32054	Quot.	91.02	1.95915

42. The *decimal* part of the logarithm may be subtracted

as in common arithmetic. But for the *indices*, when either of them is negative, or the lower one is greater than the upper one, it will be necessary to make use of the general rule for subtraction in algebra; that is, to change the signs of the subtrahend, and then proceed as in addition. (Alg. 60.) When 1 is carried from the decimal part, this is to be considered affirmative, and applied to the index, before the sign is changed.

Divide	0.8697	$\bar{1}.93937$	or	9.93937
By	98.65	1.99410		1.99410
Quot.	0.008816	$\bar{3}.94527$		7.94527

In this example, the upper logarithm being less than the lower one, it is necessary to borrow 10, as in other cases of subtraction; and therefore to carry one to the lower index, which then becomes +2. This changed to —2, and added to —1 above it, makes the index of the difference of the logarithms —3.

Divide	29.76	1.47363		1.47363
By	6254	3.79616		3.79616
Quot.	0.00476	$\bar{3}.67747$	or	7.67747

Here, 1 carried to the lower index, makes it +4. This changed to —4, and added to 1 above it, gives —3 for the index of the difference of the logarithms.

Divide	6.832	0.83455	Divide	0.00634	$\bar{3}.80209$
By	.0362	$\bar{2}.55871$	By	62.18	1.79365
Quot.	188.73	2.27584	Quot.	0.000102	$\bar{4}.00844$

The quotient of 0.0985 divided by 0.007241, is 13.6
The quotient of 0.0621 divided by 3.68, is 0.01687

43. To divide *negative* quantities, proceed in the same manner as if they were positive, (Art. 40.) and prefix to the

quotient, the sign which is required by the rules for division in algebra.

Divide	$+3642$	3.56134	Divide	-0.657	$\bar{1}.81757$
By	-23.68	1.37438	By	$+0.0793$	$\bar{2}.89927$
Quot.	-153.8	2.18696	Quot.	-8.285	0.91830

In these examples, the sign of the divisor being different from that of the dividend, the sign of the quotient must be negative. (Alg. 100.)

Divide	-0.364	$\bar{1}.56110$	Divide	-68.5	1.83569
By	-2.56	0.40824	By	$+0.094$	$\bar{2}.97313$
Quot.	$+0.1422$	$\bar{1}.15286$	Quot.	-728.7	2.86256

INVOLUTION BY LOGARITHMS.

44. Involving a quantity is multiplying it into itself. By means of logarithms, multiplication is performed by addition. If, then, the logarithm of any quantity be *added to itself*, the logarithm of a *power* of that quantity will be obtained. But adding a logarithm, or any other quantity, to itself, is *multiplication*. The involution of quantities, by means of logarithms, is therefore performed, by multiplying the logarithms.

Thus the logarithm
of 100 is 2
of 100×100, that is, of $\overline{100}^2$ is $2+2 = 2\times2$
of $100\times100\times100$, $\overline{100}^3$ is $2+2+2 = 2\times3$
of $100\times100\times100\times100$, $\overline{100}^4$ is $2+2+2+2 = 2\times4$

On the same principle, the logarithm of $\overline{100}^n$ is $2\times n$.
And the logarithm of x^n, is $(\log. x)\times n$. Hence,

45. To involve a quantity by logarithms, MULTIPLY THE LOGARITHM OF THE QUANTITY, BY THE INDEX OF THE POWER REQUIRED.

The reason of the rule is also evident, from the consideration, that logarithms are the exponents of powers and roots, and a power or root is involved, by *multiplying* its index into the index of the power required. (Alg. 170, 242.)

Ex. 1. What is the cube of		6.296 ?
Root 6.296,	its log.	0.79906
	Index of the power	3
Power 249.6		2.39718

2. Required the 4th power of		21.32
Root 21.32	log.	1.32879
	Index	4
Power 206614		5.31516

3. Required the 6th power of		1.689
Root 1.689	log.	0.22763
	Index	6
Power 23.215		1.36578

4. Required the 144th power of		1.003
Root 1.003	log.	0.00130
	Index	144
Power 1.539		0.18720

46. It must be observed, as in the case of multiplication, (Art. 38.) that what is carried from the *decimal* part of the logarithm is *positive*, whether the index itself is positive or negative. Or, if 10 be added to a negative index, to render it positive, (Art, 12.) this will be multiplied, as well as the other figures, so that the logarithm of the square, will be 20 too great; of the cube, 30 too great, &c.

Ex. 1. Required the cube of		0.0649		
Root 0.0649	log.	$\bar{2}$.81224	or	8.81224
	Index	3		3
Power 0.0002733		$\bar{4}$.43672		6.43672

2. Required the 4th power of			0.1234		
Root	0.1234	log.	$\bar{1}\,09132$	or	9.09132
		Index	4		4
Power	0.0002319		$\bar{4}.36528$		6.36528

3. Required the 6th power of			0.9977		
Root	0.9977	log.	$\bar{1}.99900$	or	9.99900
		Index	6		6
Power	0.9863		$\bar{1}.99400$		9.99400

4. Required the cube of			0.08762		
Root	0.08762	log.	$\bar{2}.94260$	or	8.94260
		Index	3		3
Power	0.0006727		$\bar{4}.82780$		6.82780

5. The 7th power of 0.9061 is 0.5015.

6. The 5th power of 0.9344 is 0.7123.

EVOLUTION BY LOGARITHMS.

47. Evolution is the opposite of involution. Therefore, as quantities are involved, by the *multiplication* of logarithms, roots are extracted by the *division* of logarithms; that is,

To extract the root of a quantity by logarithms, DIVIDE THE LOGARITHM OF THE QUANTITY, BY THE NUMBER EXPRESSING THE ROOT REQUIRED.

The reason of the rule is evident also, from the fact, that logarithms are the exponents of powers and roots, and evolution is performed, by dividing the exponent, by the number expressing the root required. (Alg. 210.)

1. Required the square root of 648.3

	Numbers.	Logarithms.
Power	648.3	2)2.81178
Root	25.46	1.40589

2. Required the cube root of 897.1

Power 897.1 3)2.95284
Root 9.645 0.98428

In the first of these examples, the logarithm of the given number is divided by 2; in the other, by 3.

3. Required the 10th root of 6948.

Power 6948 10)3.84186
Root 2.422 0.38418

4. Required the 100dth root of 983.

Power 983 100)2.99255
Root 1.071 0.02992

The division is performed here, as in other cases of decimals, by removing the decimal point to the left.

5. What is the ten thousandth root of 49680000?

Power 49680000 10000)7.69618
Root 1.00179 0.00077

We have, here, an example of the great rapidity with which arithmetical operations are performed by logarithms.

48. If the index of the logarithm is *negative,* and is *not divisible* by the given divisor, without a remainder, a difficulty will occur, unless the index be altered.

Suppose the cube root of 0.0000892 is required. The logarithm of this is $\overline{5}.95036$. If we divide the index by 3, the quotient will be —1, with —2 remainder. This remainder, if it were positive, might, as in other cases of division, be prefixed to the next figure. But the remainder is *negative,* while the decimal part of the logarithm is positive; so that, when the former is prefixed to the latter, it will make neither +2.9 nor —2.9, but —2+.9. This embarrassing intermixture of positives and negatives may be avoided, by adding to the index another negative number, to make it exactly divisible by the divisor. Thus, if to the index —5 there be added —1, the sum —6 will be divisible by 3. But this addition of a negative number must be *compensated*, by the addition of an equal positive number, which may be prefixed to the decimal part of the logarithm. The division may then be continued, without difficulty, through the whole.

Thus, if the logarithm $\bar{5}.95036$ be altered to $\bar{6}+1.95036$ it may be divided by 3, and the quotient will be $\bar{2}.65012$. We have then this rule,

49. *Add to the index*, if necessary, *such a negative number as will make it exactly divisible by the divisor, and prefix an equal positive number to the decimal part of the logarithm.*

1. Required the 5th root of 0.009642
Power 0.009642 log. $\bar{3}.98417$
or $\bar{5}+2.98417$
Root 0.3952 $\bar{1}.59683$

2. Required the 7th root of 0.0004935.
Power 0.0004935 log. $\bar{4}.69329$
or $7)\bar{7}+3.69329$
Root 0.337 $\bar{1}.52761$

4.693
3.
7. +3.

50. If, for the sake of performing the division conveniently, the negative index be rendered *positive*, it will be expedient to borrow as many tens, as there are units in the number denoting the root.

What is the fourth root of 0.03698 ?

Power 0.03698 $4)\bar{2}.56797$ or $4)38.56797$
Root 0.4385 $\bar{1}.64199$ 9.64199

Here the index, by borrowing, is made 40 too great, that is, +38 instead of —2. When, therefore, it is divided by 4, it is still 10 too great, +9 instead of —1.

What is the 5th root of 0.008926?

Power	0.008926	5)$\bar{3}$.95066	or	5)47.95066
Root	0.38916	$\bar{1}$.59013		9.59013

51. A *power of a root* may be found by first *multiplying* the logarithm of the given quantity into the index of the power, (Art. 45.) and then *dividing* the product by the number expressing the root. (Art. 47.)

1. What is the value of $(53)^{\frac{6}{7}}$, that is, the 6th power of the 7th root of 53?

Given number	53	log.	1.72428
Multiplying by			6
Dividing by			7)10.34568
Power required	30.06		1.47795

2. What is the 8th power of the 9th root of 654?

PROPORTION BY LOGARITHMS.

52. In a proportion, when three terms are given, the fourth is found in common arithmetic, by multiplying together the second and third, and dividing by the first. But when logarithms are used, *addition* takes the place of multiplication, and *subtraction*, of division.

To find, then, by logarithms, the fourth term in a proportion, ADD THE LOGARITHMS OF THE SECOND AND THIRD TERMS, AND *from the sum* SUBTRACT THE LOGARITHM OF THE FIRST TERM. The remainder will be the logarithm of the term required.

Ex. 1. Find a fourth proportional to 7964, 378, and 27960.

	Numbers.	Logarithms.
Second term	378	2.57749
Third term	27960	4.44654
		7.02403
First term	7964	3.90113
Fourth term	1327	3.12290

2. Find a 4th proportional to 768, 381, and 9780.

Second term	381	2.58092
Third term	9780	3.99034
		6.57126
First term	768	2.88536
Fourth term	4852	3.68590

ARITHMETICAL COMPLEMENT.

53. When one number is to be subtracted from another, it is often convenient, first to subtract it from 10, then to *add the difference* to the other number, and afterwards to reject the 10.

Thus, instead of $a-b$, we may put $10-b+a-10$.

In the first of these expressions, b is subtracted from a. In the other, b is subtracted from 10, the difference is added to a, and 10 is afterwards taken from the sum. The two expressions are equivalent, because they consist of the same terms, with the addition, in one of them, of $10-10=0$. The alteration is, in fact, nothing more than borrowing 10, for the sake of convenience, and then rejecting it in the result.

Instead of 10, we may borrow, as occasion requires, 100, 1000, &c.

Thus, $a-b=100-b+a-100=1000-b+a-1000$, &c.

54. The DIFFERENCE *between a given number and* 10, *or* 100, *or* 1000, *&c., is called the* ARITHMETICAL COMPLEMENT *of that number.*

The arithmetical complement of a number consisting of *one* integral figure, either with or without decimals, is found, by subtracting the number from 10. If there are *two* integral figures, they are subtracted from 100; if *three,* from 1000, &c.

Thus, the arithmetical compl't of 3.46 is $10-3.46=6.54$
of 34.6 is $100-34.6=65.4$
of 346. is $1000-346.=654.$ &c.

According to the rule for subtraction in arithmetic, any number is subtracted from 10, 100, 1000, &c. by beginning on the right hand, and taking each figure from 10, after *increasing* all except the first, by *carrying* 1.

Thus, if from 10.00000
We subtract 7.63125

The difference, or arith'l compl't is 2.36875, which is obtained by taking 5 from 10, 3 from 10, 2 from 10, 4 from 10, 7 from 10, and 8 from 10. But, instead of taking each figure, *increased by* 1 from 10; we may take it *without being increased,* from 9.

Thus, 2 from 9 is the same as 3 from 10,
3 from 9 the same as 4 from 10, &c. Hence,

55. *To obtain the* ARITHMETICAL COMPLEMENT *of a number, subtract the right hand significant figure from* 10, *and each of the other figures from* 9. If, however, there are *ciphers* on the right hand of all the significant figures, they are to be set down without alteration.

In taking the arithmetical complement of a logarithm, if the index is *negative,* it must be *added* to 9; for adding a negative quantity is the same as subtracting a positive one. (Alg. 81.) The difference between —3 and +9, is not 6, but 12.

The arithmetical complement

of 6.24897 is	3.75103	of $\bar{2}$.70649 is	11.29351
of 2.98643	7.01357	of 3.64200	6.35800
of 0.62430	9.37570	of 9.35001	0.64999

56. The principal use of the arithmetical complement, is in working proportions by logarithms; where some of the terms are to be *added,* and one or more to be *subtracted.* In the Rule of Three or simple proportion, two terms are to be added, and from the sum, the first term is to be subtracted. But if, instead of the logarithm of the first term,

we substitute its arithmetical complement, this may be *added* to the sum of the other two, or more simply all three may be added together, by one operation. After the index is diminished by 10, the result will be the same as by the common method. For subtracting a number is the same, as adding its arithmetical complement, and then rejecting 10, 100, or 1000, from the sum. (Art. 53.)

It will generally be expedient, to place the terms in the same order, in which they are arranged in the statement of the proportion.

1.	As	6273	*a. c.*	6.20252
	Is to	769.4		2.88615
	So is	37.61		1.57530
	To	4.613		0.66397

2.	As	253	*a. c.*	7.59688
	Is to	672.5		2.82769
	So is	497		2.69636
	To	1321.1		3.12093

3.	As	46.34	*a. c.*	8.33404
	Is to	892.1		2.95041
	So is	7.638		0.88298
	To	147		2.16743

4.	As	9.85	*a. c.*	9.00656
	Is to	643		2.80821
	So is	76.3		1.88252
	To	4981		3.69729

COMPOUND PROPORTION.

57. In compound, as in single proportion, the term required may be found by logarithms, if we substitute addition for multiplication, and subtraction for division.

Ex. 1. If the interest of $365, for 3 years and 9 months, be $82.13; what will be the interest of $8940, for 2 years and 6 months?

In common arithmetic, the statement of the question is made in this manner.

$$\left.\begin{array}{ll}365 & \text{dollars}\\ 3.75 & \text{years}\end{array}\right\} : \left\{\begin{array}{ll}8940 & \text{dollars}\\ 2.5 & \text{years}\end{array}\right\} :: 82.13 \text{ dollars} :$$

And the method of calculation is, to *divide* the *product* of the third, fourth, and fifth terms, by the *product* of the first two.* This, if logarithms are used, will be to *subtract* the *sum* of the logarithms of the first two terms, from the *sum* of the logarithms of the other three.

First two terms	365 log.	2.56229
	3.75	0.57403
Sum of the logarithms		3.13632
Third and fourth terms	8940	3.95134
	2.5	0.39794
Fifth term	82.13	1.91450
Sum of the logs. of the 3rd, 4th, and 5th,		6.26378
Do.	1st and 2nd,	3.13632
Term required	1341	3.12746

58. The calculation will be more simple, if, instead of *subtracting* the logarithms of the first two terms, we *add* their *arithmetical complements*. But, it must be observed, that *each* arithmetical complement increases the index of the logarithm by 10. If the arithmetical complement be introduced into *two* of the terms, the index of the sum of the logarithms will be 20 too great; if it be in *three* terms, the index will be 30 too great, &c.

First two terms	365	*a. c.*	7.43771
	3.75	*a. c.*	9.42597
Third and fourth terms	8940		3.95134
	2.5		0.39794
Fifth term	82.13		1.91450
Term required	1341		23.12746

The result is the same as before, except that the index of the logarithm is 20 too great.

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* See Arithmetic.

Ex. 2. If the wages of 53 men for 42 days be 2200 dollars; what will be the wages of 87 men for 34 days?

$$\left.\begin{matrix}53\text{ men}\\ 42\text{ days}\end{matrix}\right\} : \left\{\begin{matrix}87\text{ men}\\ 34\text{ days}\end{matrix}\right\} :: 2200 :$$

First two terms	53. *a. c.*	8.27572
	42. *a. c.*	8.37675
Third and fourth terms	87	1.93952
	34	1.53148
Fifth term	2200	3.34242
Term required	2923.5	3.46589

59. In the same manner, if the product of *any number* of quantities, is to be divided, by the product of several others; we may add together the logarithms of the quantities to be divided, and the arithmetical complements of the logarithms of the divisors.

Ex. If 29.67×346.2 be divided by 69.24×7.862×497; what will be the quotient?

Numbers to be divided	29.67	1.47232
	346.2	2.53933
Divisors	69.24 *a. c.*	8.15964
	7.862 *a. c.*	9.10447
	497 *a. c.*	7.30364
Quotient	0.03797	8.5794

In this way, the calculations in *Conjoined Proportion* may be expeditiously performed.

COMPOUND INTEREST.

60. In calculating compound interest, the amount for the first year, is made the principal for the second year; the amount for the second year, the principal for the third

year, &c. Now the amount at the end of each year, must be proportioned to the principal at the beginning of the year. If the principal for the first year be 1 dollar, and if the amount of 1 dollar for 1 year$=a$; then, (Alg. 341.)

$$1 : a :: \begin{cases} a : a^2 = \text{the amount for the 2d year, or the principal for the 3d;} \\ a^2 : a^3 = \text{the amount for the third year, or the principal for the 4th;} \\ a^3 : a^4 = \text{the amount for the 4th year, or the principal for the 5th.} \end{cases}$$

That is, the amount of 1 dollar for any number of years is obtained by finding the amount for 1 year, and involving this to a power whose index is equal to the number of years. And the amount of any other principal, for the given time, is found by multiplying the amount of 1 dollar, into the number of dollars, or the fractional part of a dollar.

If logarithms are used, the *multiplication* required here may be performed by *addition*; and the *involution* by *multiplication*. (Art. 45.) Hence,

61. To calculate Compound Interest, *Find the amount of* 1 *dollar for* 1 *year; multiply its logarithm by the number of years; and to the product, add the logarithm of the principal.* The sum will be the logarithm of the *amount* for the given time. From the amount subtract the principal, and the remainder will be the *interest.*

If the interest becomes due *half yearly* or *quarterly*; find the amount of one dollar, for the half year or quarter, and multiply the logarithm by the number of half years or quarters in the given time.

If $P=$the principal,

$a=$the amount of 1 dollar for 1 year,

$n=$any number of years, and

$A=$the amount of the given principal for n years; then,

$$A=a^n\times P.$$

Taking the logarithms of both sides of the equation, and reducing it, so as to give the value of each of the four quantities, in terms of the others, we have

$$1.\quad \text{Log. } A = n \times \text{log. } a + \text{log. } P.$$

$$2.\quad \text{Log. } P = \text{log. } A - n \times \text{log. } a.$$

$$3.\quad \text{Log. } a = \frac{\text{log. } A - \text{log. } P.}{n}$$

$$4.\quad n = \frac{\text{log. } A - \text{log. } P.}{\text{log } a.}$$

Any three of these quantities being given, the fourth may be found.

Ex. 1. What is the amount of 20 dollars, at 6 per cent. compound interest, for 100 years?

Amount of 1 dollar for 1 year	1.06	log.	0.0253059
Multiplying by			100
			2.53059
Given principal	20		1.30103
Amount required	\$6786		3.83162

2. What is the amount of 1 cent at 6 per cent. compound interest, in 500 years?

Amount of 1 dollar for 1 year	1.06	log.	0.0253059
Multiplying by			500
			12.65295
Given principal	0.01		–2.00000
Amount	\$44,973,000,000		10.65295

More exact answers may be obtained, by using logarithms of a greater number of decimal places.

3. What is the amount of 1000 dollars, at 6 per cent. compound interest, for 10 years? Ans. 1790.80.

4. What principal, at 4 per cent. interest, will amount to 1643 dollars in 21 years? Ans. 721.

5. What principal, at 6 per cent., will amount to 202 dollars in 4 years? Ans. 160.

6. At what rate of interest, will 400 dollars amount to 569⅓, in 9 years? Ans. 4 per cent.

7. In how many years will 500 dollars amount to 900, at 5 per cent. compound interest? Ans. 12 years.

8. In what time will 10,000 dollars amount to 16,288, at 5 per cent compound interest? Ans. 10 years.

9. At what rate of interest, will 11,106 dollars amount to 20.000 in 15 years? Ans. 4 per cent.

10. What principal, at 6 per cent. compound interest, will amount to 3188 dollars in 8 years? Ans. $2000.

11. What will be the amount of 1200 dollars, at 6 per cent compound interest, in 10 years, if the interest is converted into principal every *half year?* Ans. 2167.3 dolls.

12. In what time will a sum of money double, at 6 per cent compound interest? Ans. 11.9 years.

13. What is the amount of 5000 dollars, at 6 per cent. compound interest, for 28¼ years? Ans. 25.942 dollars.

INCREASE OF POPULATION.

62. The natural increase of population in a country, may be calculated in the same manner as compound interest; on the supposition, that the yearly rate of increase is regularly proportioned to the actual number of inhabitants. From the population at the beginning of the year, the *rate* of increase being given, may be computed the whole increase during the year. This, *added* to the number at the beginning, will give the amount, on which the increase of the *second* year is to be calculated, in the same manner as the

first year's interest on a sum of money, added to the sum itself, gives the amount on which the interest for the second year is to be calculated.

If P=the population at the beginning of the year,
$a=1+$the fraction which expresses the rate of increase,
n=any number of years; and
A=the amount of the population at the end of n years;
then, as in the preceding article,

$$A=a^n\times P, \text{ and}$$

$$1.\ \text{Log. } A=n\times\text{log. } a+\text{log. } P.$$

$$2.\ \text{Log. } P=\text{log. } A-n\times\text{log. } a.$$

$$3.\ \text{Log. } a=\frac{\text{log. } A-\text{log. } P.}{n}$$

$$4.\ n=\frac{\text{log. } A-\text{log. } P.}{\text{log. } a.}$$

Ex. 1. The population of the United States in 1840 was (in round numbers) 17,070,000.* Supposing the yearly rate

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* For some very interesting views of the progress of population, &c., in the United States, see Prof. George Tucker's elaborate essays, first published in the Merchant's Magazine, 1842—3, and subsequently in a separate volume.

The following tables show the official census of the United States from 1790 to 1840 with the decennial rate of increase.

POPULATION.					
1790.	1800.	1810.	1820.	1830.	1840.
3,929,827	5,305,925	7,239,814	9,638,131	12,866,020	17,069,453

DECENNIAL INCREASE.				
1800.	1810.	1820.	1830.	1840.
35.02	36.45	33.35	33.26	33.67

The population of the U. States in 1850 was 23.191.876.

of increase to be $\frac{1}{34}$ part of the whole, what will be the population in 1850?

Here $P=17,070,000$. $n=10$. $a=1+\frac{1}{34}=\frac{35}{34}$.

And $\log. A=10\times\log. \frac{35}{34}+\log. (17,070,000,)$

Therefore, $A=22,810,000$, the population in 1850.

ctual pop in 1850. 23,191,876

2. If the number of inhabitants in a country be five millions at the beginning of a century; and if the yearly rate of increase be $\frac{1}{30}$; what will be the number at the end of 50 years? and what at the end of the century?

Ans. 25,763,000, and 132,750,000.

3. If the population of a country, at the end of a century, is found to be 45,860,000; and if the yearly rate of increase has been $\frac{1}{120}$; what was the population at the commencement of the century? Ans. 20 millions.

4. The population of the United States in 1810 was 7,240,000; in 1820, 9,625,000. What was the annual rate of increase between these two periods, supposing the increase each year to be proportioned to the population at the beginning of the year?

$$\text{Here } \log. a=\frac{\log. 9,625,000-\log. 7,240,000}{10}$$

Therefore, $a=1.029$; and $\frac{29}{1000}$, or 2.9 per cent. is the rate of increase.

5. The population of the United States on the 1st August, 1820, was 9,638,000—in 1830, the time of taking the census was changed to the 1st June, and at that time the population was 12,866,000.—What was the annual rate of increase? And what would have been the amount of population to be added for the subsequent two months?

6. In how many years, will the population of a country advance from two millions to five millions; supposing the yearly rate of increase to be $\frac{7}{360}$? Ans. $47\frac{1}{2}$ years.

7. If the population of a country, at a given time, be seven millions; and if the yearly rate of increase be $\frac{1}{25}$th; what will be the population at the end of 35 years?

8. The population of the United States in 1800 was 5,306,000. What was it in 1780, supposing the yearly rate of increase to be $\frac{1}{25}$?

9. In what time will the population of a country advance, from four millions to seven millions, if the ratio of increase be $\frac{3}{100}$?

10. What must be the rate of increase, that the population of a place may change from nine thousand to fifteen thousand, in 12 years?

If the population of a country is not affected by immigration or emigration, the rate of increase will be equal to the difference between the ratio of the *births*, and the ratio of the *deaths*, when compared with the whole population.

Ex. 11. If the population of a country, at any given time, be ten millions; and the ratio of the annual number of births to the whole population be $\frac{1}{24}$, and the ratio of deaths $\frac{1}{45}$, what will be the number of inhabitants, at the end of 60 years?

Here the yearly rate of increase $=\frac{1}{24}-\frac{1}{45}=\frac{7}{360}$.

And the population, at the end of 60 years $=31,750,000$.

The rate of increase or decrease from *immigration* or *emigration*, will be equal to the difference between the ratio of immigration and the ratio of emigration; and if this differ- be added to, or subtracted from, the difference between the ratio of the births and that of the deaths, the whole rate of increase will be obtained.

Ex. 12. If in a country, the ratio of births be $\frac{1}{30}$,
the ratio of deaths $\frac{1}{40}$,
the ratio of immigration $\frac{1}{50}$,
the ratio of emigration $\frac{1}{60}$,

and if the population this year be 10 millions, what will it be 20 years hence?

The rate of the natural increase $=\frac{1}{30}-\frac{1}{40}=\frac{1}{120}$;
That of increase from immigration $=\frac{1}{50}-\frac{1}{60}=\frac{1}{300}$;
The sum of the two is $\frac{7}{600}$;
And the population at the end of 20 years, is 12,611,000.

13. If the ratio of the births be $\frac{1}{20}$,
of the deaths $\frac{1}{30}$,
of immigration $\frac{1}{40}$,
of emigration $\frac{1}{50}$,
in what time will three millions increase to four and a half millions?

If the period in which the population will *double* be given; the numbers for several successive periods, will evidently be in a geometrical progression, of which the ratio is 2; and as the number of periods will be one less than the number of terms;

If P=the first term,
A=the last term,
n=the number of periods;
Then will $A=P\times 2^n$, (Alg. 439.)
Or log. A=log. $P+n\times$log. 2.

Ex 1. If the descendants of a single pair double once in 25 years, what will be their number at the end of one thousand years?

The number of periods here is 40.
And $A=2\times 2^{40}=2{,}199{,}200{,}000{,}000$.

2. If the descendants of Noah, beginning with his three sons and their wives, doubled once in 20 years for 300 years, what was their number, at the end of this time?

Ans. 196,608.

3. The population of the United States in 1820 being

9,638,000; what must it be in the year 2020, supposing it to double once in 25 years? Ans. 2,467,333,000.

4. Supposing the descendants of the first human pair to double once in 50 years, for 1650 years, to the time of the deluge, what was the population of the world, at that time?

EXPONENTIAL EQUATIONS.

62. An EXPONENTIAL equation is one in which the letter expressing the unknown quantity is an *exponent.*

Thus $a^x=b$, and $x^x=bc$, are exponential equations. These are most easily solved by logarithms. As the two members of an equation are equal, their logarithms must also be equal. If the logarithm of each side be taken, the equation may then be reduced, by the rules given in algebra.

Ex. What is the value of x in the equation $3^x=243$?

Taking the logarithms of both sides, log. 3^x=log. 243.

But the logarithm of a *power* is equal to the logarithm of the root, multiplied into the index of the power. (Art. 45.)

Therefore (log. 3)$\times x$=log. 243; and dividing by log. 3.

$$x=\frac{\text{log. } 243}{\text{log } 3.}=\frac{2.38561}{0.47712}=5. \quad \text{So that } 3^5=243.$$

64. The exponent of a power may be itself a power, as in the equation

$$a^{m^x}=b;$$

where x is the exponent of the power m^x, which is the exponent of the power a^{m^x}.

Ex. 4. Find the value of x, in the equation $9^{3^x}=1000$.

$$3^x\times(\text{log. } 9)=\text{log. } 1000. \quad \text{Therefore, } 3^x=\frac{\text{log. } 1000.}{\text{log. } 9}=3.14.$$

Then, as $3^x=3.14$. $x(\log. 3)=\log. 3.14$

$$\text{Therefore, } x=\frac{\log. 3.14}{\log. 3}=\tfrac{4969296}{4771213}=1.04.$$

In cases like this, where the factors, divisors, &c. are logarithms, the calculation may be facilitated, by taking the *logarithms of the logarithms.* Thus the value of the fraction $\frac{4969296}{4771213}$ is most easily found, by subtracting the logarithm of the logarithm which constitutes the denominator, from the logarithm of that which forms the numerator.

5. Find the value of x, in the equation $\dfrac{ba^x+d}{c}=m$

$$\text{Ans. } x=\frac{\log. (cm-d)-\log. b.}{\log. a.}$$

TRIGONOMETRY.

SECTION I.

SINES, TANGENTS, SECANTS, &C.

ART. 71. TRIGONOMETRY *treats of the relations of the sides and angles of* TRIANGLES. Its first object is to determine the length of the sides, and the quantity of the angles. In addition to this, from its principles are derived many interesting methods of investigation in the higher branches of analysis, particularly in physical astronomy.

72. Trigonometry is either *plane* or *spherical*. The former treats of triangles bounded by *right lines;* the latter, of triangles bounded by *arcs of circles.*

Divisions of the Circle.

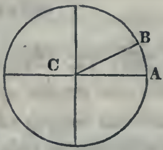
73. In a triangle there are two classes of quantities which are the subjects of inquiry, the *sides* and the *angles*. For the purpose of measuring the latter, a *circle* is introduced.

The periphery of every circle, whether great or small, is supposed to be divided into 360 equal parts called *degrees*, each degree into 60 *minutes*, each minute into 60 *seconds*, each second into 60 *thirds*, &c., marked with the characters $^\circ$, $'$, $''$, $'''$, &c. Thus, $32^\circ\ 24'\ 13''\ 22'''$ is 32 degrees, 24 minutes, 13 seconds, 22 thirds.

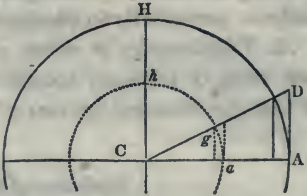
A degree, then, is not a magnitude of a given *length;* but a certain *portion* of the whole circumference of any circle. It is evident that the 360th part of a large circle is greater than the same part of a small one. On the other hand, the *number* of degrees in a small circle, is the same as in a large one.

The fourth part of a circle is called a *quadrant*, and contains 90 degrees.

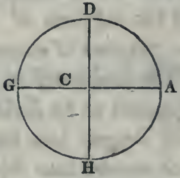
74. To *measure* an angle, a circle is so described that its center shall be the angular point, and its periphery shall cut the two lines which include the angle. The *arc* between the two lines is considered a *measure of the angle*, because, by Euc. 33. 6, angles at the center of a given circle, have the same ratio to each other, as the arcs on which they stand. Thus the arc AB, is a measure of the angle ACB.



It is immaterial what is the size of the circle, provided it cuts the lines which include the angle. Thus, the angle ACD is measured by either of the arcs AG, *ag*. For ACD is to ACH, as AG to AH, or as *ag* to *ah*. (Euc. 33. 6.)



75. In the circle ADGH, let the two diameters AG and DH be perpendicular to each other. The angles ACD, DCG, GCH, and HCA, will be right angles; and the periphery of the circle will be divided into four equal parts, each containing 90 degrees. As a right angle is subtended by an arc of 90°, the angle itself is said to contain 90°. Hence, in two



right angles, there are 180°; in four right angles, 360°; and in any other angle, as many degrees, as in the arc by which it is subtended.

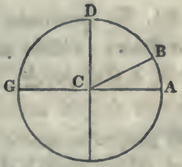
76. The sum of the three angles of any triangle being equal to two right angles, (Euc. 32. 1.*) is equal to 180°. Hence, there can never be more than one obtuse angle in a triangle. For the sum of two obtuse angles is more than 180°.

77. *The* COMPLEMENT *of an arc or an angle, is the difference between the arc or angle and* 90 *degrees.*

The complement of the arc AB is DB; and the complement of the angle ACB is DCB. The complement of the arc BDG is also DB.

The complement of 10° is 80°, of 60° is 30°,
of 20° is 70°, of 120° is 30°,
of 50° is 40°, of 170° is 80°, &c.

Hence, an acute angle and its complement are always equal to 90°. The angles ACB and DCB are together equal to a right angle. The two acute angles of a right angled triangle are equal to 90°: therefore each is the complement of the other.



78. *The* SUPPLEMENT *of an arc or an angle is the difference between the arc or angle and* 180 *degrees.*

The supplement of the arc BDG is AB; and the supplement of the angle BCG is BCA.

The supplement of 10° is 170°, of 120° is 60°,
of 80° is 100°, of 150° is 30°, &c.

Hence an angle and its supplement are always equal to

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* Thomson's Geometry, 28, 1.

180°. The angles BCA and BCG are together equal to two right angles.

79. Cor. As the three angles of a plane triangle are equal to two right angles, that is, to 180° (Euc. 32. 1.) the sum of any two of them is the supplement of the other. So that the third angle may be found, by subtracting the sum of the other two from 180°. Or the sum of any two may be found, by subtracting the third from 180°.

80. A straight line drawn from the centre of a circle to any part of the periphery, is called a *radius* of the circle. In many calculations, it is convenient to consider the *radius*, whatever be its length, as *a unit.* (Alg. 510.) To this must be referred the numbers expressing the lengths of other lines. Thus, 20 will be twenty times the radius, and 0.75, three-fourths of the radius.

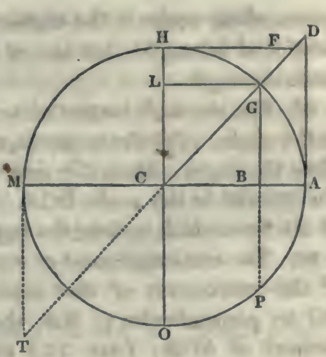
Definitions of Sines, Tangents, Secants, &c.

81. To facilitate the calculations in Trigonometry, there are drawn, within and about the circle, a number of straight lines, called *Sines, Tangents, Secants, &c.* With these the learner should make himself perfectly familiar. The direct and proper measure of an angle is an *arc* of a circle. (Art. 74.) But trigonometrical solutions are commonly made with the aid of certain *straight* lines, which have known relations to the arcs to which they belong.

82. *The* SINE *of an arc is a straight line drawn from one end of the arc, perpendicular to a diameter which passes through the other end.*

Thus, BG is the sine of the arc AG. For BG is a line drawn from the end G of the arc, perpendicular to the diameter AM which passes through the other end A of the arc.

Cor. The sine is *half the chord* of *double the arc.* The sine BG is half PG, which is the chord of the arc PAG, double the arc AG.



83. *The* VERSED SINE *of an arc is that part of the diameter which is between the sine and the arc.*

Thus, BA is the versed sine of the arc AG.

84. *The* TANGENT *of an arc, is a straight line drawn perpendicularly from the extremity of the diameter which passes through one end of the arc, and extended till it meets a line drawn from the centre through the other end.*

Thus, AD is the tangent of the arc AG.

85. *The* SECANT *of an arc is a straight line drawn from the centre, through one end of the arc, and extended to the tangent which is drawn from the other end.*

Thus CD is the secant of the arc AG.

86. In Trigonometry, the terms *tangent* and *secant* have a more limited meaning, than in Geometry. In both, indeed, the tangent *touches* the circle, and the secant *cuts* it. But in Geometry, these lines are of no determinate length; whereas, in Trigonometry, they extend from the diameter to the point in which they intersect each other.

87. The lines just defined are sines, tangents, and secants of *arcs.* BG is the sine of the arc AG. But this arc subtends the *angle* GCA. BG is then the sine of the arc which subtends the *angle* GCA. This is more concisely expressed, by saying that BG is the sine of the angle GCA. And universally, the sine, tangent, and secant of an *arc,* are said to be the sine, tangent, and secant of the *angle* which stands at the centre of the circle, and is subtended by the arc. Whenever, therefore, the sine, tangent, or secant of an angle is spoken of; we are to suppose a circle to be

drawn whose centre is the angular point; and that the lines mentioned belong to that arc of the periphery which subtends the angle.

88. The *sine* and *tangent* of an acute angle, are *opposite* to the angle. But the *secant* is one of the lines which *include* the angle. Thus, the sine BG, and the tangent AD, are opposite to the angle DCA. But the secant CD is one of the lines which include the angle.

89. *The sine complement or* COSINE *of an angle, is the sine of the* COMPLEMENT *of that angle.* Thus, if the diameter HO be perpendicular to MA, the angle HCG is the complement of ACG; (Art. 77.) and LG, or its equal CB, is the sine of HCG. (Art. 82.) It is, therefore, the *cosine* of GCA. On the other hand, GB is the sine of GCA, and the cosine of GCH.

So also the *cotangent* of an angle is the tangent of the *complement* of the angle. Thus, HF is the cotangent of GCA. And the *cosecant* of an angle is the secant of the *complement* of the angle. Thus, CF is the cosecant of GCA.

Hence, as in a right angled triangle, one of the acute angles is the complement of the other; (Art. 77.) the sine, tangent, and secant of one of these angles, are the cosine, cotangent, and cosecant of the other.

90. The sine, tangent, and secant of the *supplement* of an angle, are each equal to the sine, tangent, and secant of the angle itself. It will be seen, by applying the definition. (Art 82.) to the figure, that the sine of the obtuse angle GCM is BG, which is also the sine of the acute angle GCA. It should be observed, however, that the sine of an acute angle is *opposite* to it; while the sine of an obtuse angle *falls without* the angle, and is opposite to its supplement. Thus BG, the sine of the angle MCG, is not opposite to MCG, but to its supplement ACG.

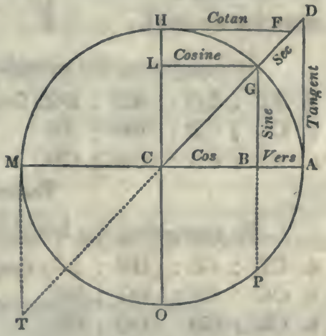
The *tangent* of the obtuse angle MCG is MT, or its equal

AD, which is also the tangent of ACG. And the *secant* of MCG is CD, which is also the secant of ACG.

91. But the *versed sine* of an angle *is not the same* as that of its *supplement.* The versed sine of an *acute* angle is equal to the *difference* between the cosine and radius. But the versed sine of an *obtuse* angle is equal to the *sum* of the cosine and radius. Thus, the versed sine of ACG is AB=AC—BC. (Art. 83.) But the versed sine of MCG is MB=MC+BC.

Relations of Sines, Tangents, Secants, &c., to each other.

92. The relations of the sine, tangent, secant, cosine, &c., to each other, are easily derived from the proportions of the sides of similar triangles. (Euc. 4. 6.*) In the quadrant ACH, these lines form three similar triangles, viz. ACD, BCG or LCG, and HCF. For, in each of these, there is one right angle, because the sines and tangents are, by definition, perpendicular to AC; as the cosine and cotangent are to CH. The lines CH, BG, and AD, are parallel, because CA makes a right angle with each. (Euc. 27. 1.†) For the same reason, CA, LG, and HF, are parallel. The alternate angles GCL, BGC, and the opposite angle CDA, are equal; (Euc. 29. 1.‡) as are also the angles GCB, LGC, and HFC. The triangles ACD, BCG, and HCF, are therefore similar.



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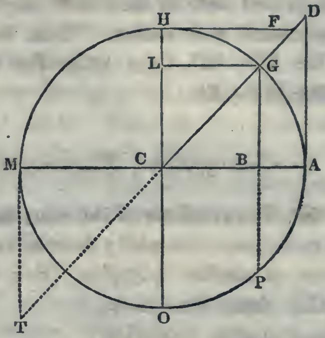
* Thomson, 18. 4. † Ibid. 18. 1. ‡ Ibid. 24. 1.

It should also be observed, that the line BC, between the sine and the centre of the circle, is parallel and equal to the cosine; and that LC, between the cosine and centre, is parallel and equal to the sine; (Euc. 34. 1.*) so that one may be taken for the other in any calculation.



93. From these similar triangles, are derived the following proportions; in which R is put for radius,

sin for sine,
cos for cosine,
tan for tangent,
cot for cotangent,
sec for secant,
cosec for cosecant.

By comparing the triangles CBG and CAD,

1. AC : BC : : AD : BG, that is, R : cos : : tan : sin.
2. CG : CD : : BG : AD R : sec : : sin : tan.
3. CB : CA : : CG : CD cos : R : : R : sec.

Therefore $R^2=\cos\times\sec$.

By comparing the triangles CLG and CHF,

4. CH : CL : : HF : LG, that is, R : sin : : cot : cos.
5. CG : CF : : LG : HF, R : cosec : : cos : cot.
6. CL : CH : : CG : CF sin : R : : R : cosec.

Therefore $R^2=\sin\times\text{cosec}$.

By comparing the triangles CAD and CHF,

7. CH : AD : : CF : CD, that is, R : tan : : cosec : sec.
8. CA : HF : : CD : CF R : cot : : sec : cosec.
9. AD : AC : : CH : HF tan : R : : R : cot.

Therefore $R^2=\tan\times\cot$.

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* Thomson, 26. 1.

It will not be necessary for the learner to commit these proportions to memory. But he ought to make himself so familiar with the manner of stating them from the figure, as to be able to explain them, whenever they are referred to.

94. Other relations of the sine, tangent, &c., may be derived from the proposition, that the square of the hypothenuse is equal to the sum of the squares of the perpendicular sides. (Euc. 47. 1.—Thomson 11. 4.)

In the right angled triangles CBG, CAD, and CHF,

1. $\overline{CG}^2=\overline{CB}^2+\overline{BG}^2$, that is, $R^2=\cos^2+\sin^2$,*
2. $\overline{CD}^2=\overline{CA}^2+\overline{AD}^2$ $\quad \sec^2=R^2+\tan^2$,
3. $\overline{CF}^2=\overline{CH}^2+\overline{HF}^2$ $\quad \mathrm{cosec}^2=R^2+\cot^2$,

And, extracting the root of both sides, (Alg. 296.)

$$R=\sqrt{\cos^2+\sin^2}=\sqrt{\sec^2-\tan^2}=\sqrt{\mathrm{cosec}^2-\cot^2}$$

Hence, if R=1, (Alg 385.)

$$\mathrm{Sin}=\sqrt{1-\cos^2} \qquad \mathrm{Sec}=\sqrt{1+\tan^2}$$
$$\mathrm{Cos}=\sqrt{1-\sin^2} \qquad \mathrm{Cosec}=\sqrt{1+\cot^2}$$

95. $\left.\begin{array}{r}\textit{The sine of } 90^\circ\\ \textit{The chord of } 60^\circ\\ \textit{And the tangent of } 45^\circ\end{array}\right\}$ *are,* in any circle, *each equal to the radius,* and therefore equal to each other.

Demonstration.

1. In the quadrant ACH, (figure on the next page,) the arc AH is 90°. The sine of this, according to the definition, (Art. 82.) is CH, the radius of the circle.

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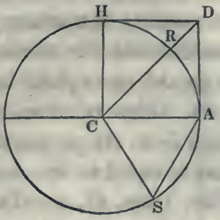
* Sin² is here put for the square of the sine, cos² for the square of the cosine, &c.



2. Let AS be an arc of 60°. Then the angle ACS, being measured by this arc, will also contain 60°; (Art. 75.) and the triangle ACS will be equilateral. For the sum of the three angles is equal to 180°. (Art. 76.) From this, taking the angle ACS, which is 60°, the sum of the remaining two is 120°. But these two are *equal*, because they are subtended by the equal sides, CA and CS, both radii of the circle. Each, therefore, is equal to *half* 120°, that is, to 60°. All the angles being equal, the sides are equal, and therefore AS, the chord of 60°, is equal to CS, the radius.

3. Let AR be an arc of 45°. AD will be its tangent, and the angle ACD subtended by the arc, will contain 45°. The angle CAD is a right angle, because the tangent is, by definition, perpendicular to the radius AC. (Art. 84.) Subtracting ACD, which is 45°, from 90°, (Art. 77.) the other acute angle ADC will be 45° also. Therefore the two legs of the triangle ACD are equal, because they are subtended by equal angles; (Euc. 6. 1.) that is, AD the tangent of 45°, is equal to AC the radius.

Cor. The *cotangent* of 45° is also equal to radius. For the complement of 45° is itself 45°. Thus, HD, the cotangent of ACD, is equal to AC the radius.

96. The sine of 30° is equal to *half radius.* For the sine of 30° is equal to half the chord of 60°. (Art. 82. cor.) But by the preceding article, the chord of 60° is equal to radius. Its half, therefore, which is the sine of 30°, is equal to half radius.

Cor. 1. The *cosine* of 60° is equal to half radius. For the cosine of 60° is the sine of 30°. (Art. 89.)

Cor. 2. The cosine of $30°=\frac{1}{2}\sqrt{3}$. For

$$\text{Cos}^2\,30°=R^2-\sin^2 30°=1-\tfrac{1}{4}=\tfrac{3}{4}.$$

Therefore,

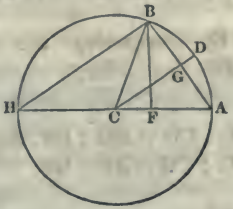
$$\text{Cos } 30^\circ = \sqrt{\tfrac{3}{4}} = \tfrac{1}{2}\sqrt{3}.$$

96. *b.* The sine of $45^\circ = \frac{1}{\sqrt{2}}$. For

$$R^2 = 1 = \sin^2 45^\circ + \cos^2 45 = 2 \sin^2 45^\circ$$

Therefore, $\text{Sin } 45^\circ = \sqrt{\tfrac{1}{2}} = \frac{1}{\sqrt{2}}$

97. The *chord* of any arc is a *mean proportional*, between the *diameter* of the circle, and the *versed sine* of the arc.



Let ADB, be an arc, of which AB is the chord, BF the sine, and AF the versed sine. The angle ABH is a right angle, (Euc. 31. 3.[*]) and the triangles ABH, and ABF, are similar. (Euc. 8. 6.[†]) Therefore,

$$AH : AB :: AB : AF.$$

That is, the diameter is to the chord, as the chord to the versed sine.

Let the arc $AD = a$, and $ADB = 2a$. Draw BF perpendicular to AH. This will divide the right angled triangle ABH into two similar triangles. (Euc. 8. 6.) The angles ACD and AHB are equal. (Euc. 20. 3.[‡]) Therefore the four triangles ACG, AHB, FHB, and FAB are similar; and the line BH is twice CG, because $BH : CG :: HA : CA$.

The sides of the four triangles are,

$AG = \sin a$, $CG = \cos a$. $HF = \text{vers. sup. } 2a$,

$AB = 2 \sin a$, $BH = 2 \cos a$. $AC =$ the radius,

$BF = \sin 2a$, $AF = \text{vers } 2a$, $AH =$ the diameter.

---

[*] Thomson, 13. 2. Cor. 2. [†] Ibid. 22. 4. [‡] Ibid. 13. 2.

A variety of proportions may be stated, between the homologous sides of these triangles: For instance,

By comparing the triangles ACG and ABF,

AC : AG : : AB : AF, that is, R : sin a : : 2 sin a : vers $2a$

AC : CG : : AB : BF, R : cos a : : 2 sin a : sin $2a$

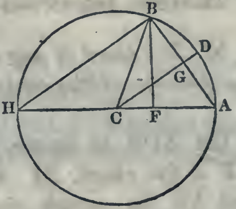
AG : CG : : AF : BF, Sin a : cos a : : vers $2a$: sin $2a$

Therefore,

$R \times \text{vers } 2a = 2\sin^2 a$

$R \times \sin 2a = 2\sin a \times \cos a$

$\text{Sin } a \times \sin 2a = \text{vers } 2a \times \cos a$



By comparing the triangles ACG and BFH,

AC : CG : : BH : HF, that is, R : cos a : : 2cos a : vers. sup. $2a$

AG : CG : : BF : HF, Sin a : cos a : : sin $2a$: vers. sup. $2a$

Therefore,

$R \times \text{vers. sup. } 2a = 2\cos^2 a$

$\text{Sin } a \times \text{vers. sup. } 2a = \cos a \times \sin 2a$

&c. &c.

That is, the product of radius into the versed sine of the supplement of twice a given arc, is equal to twice the square of the cosine of the arc.

And the product of the sine of an arc, into the versed sine of the supplement of twice the arc, is equal to the product of the cosine of the arc, into the sine of twice the arc, &c., &c.

SECTION II.

THE TRIGONOMETRICAL TABLES.

Art. 98. To facilitate the operations in trigonometry, the sine, tangent, secant, &c., have been calculated for every degree and minute, and in some instances, for every second, of a quadrant, and arranged in tables. These constitute what is called the *Trigonometrical Canon*. It is not necessary to extend these tables beyond 90°; because the sines, tangents, and secants, are of the same magnitude, in one of the quadrants of a circle, as in the others. Thus the sine of 30° is equal to that of 150°. (Art. 90.)

99. And in any instance, if we have occasion for the sine, tangent, or secant of an *obtuse angle*, we may obtain it, by looking for its equal, the sine, tangent, or secant of the *supplementary acute angle.*

100. The tables are calculated for a circle whose radius is supposed to be a *unit.* It may be an inch, a yard, a mile, or any other denomination of length. But the *sines, tangents, &c.*, must always be understood to be of the same denomination as the radius.

101. All the *sines*, except that of 90°, are *less than radius*, (Art. 82.) and are expressed in the tables by decimals.

Thus the sine of 20° is 0.34202, of 60° is 0.86603,
of 40° is 0.64279, of 89° is 0.99985, &c.

When the tables are intended to be very exact, the decimal is carried to a greater number of places.

The *tangents* of all angles less than 45° are also less than radius. (Art. 95.) But the tangents of angles greater than 45°, are *greater* than radius, and are expressed by a whole number and a decimal. It is evident that all the *secants* also must be greater than radius, as they extend from the centre, to a point without the circle.

102. The numbers in the table here spoken of, are called *natural* sines, tangents, &c. They express the lengths of the several lines which have been defined in Arts. 82, 83, &c. By means of them, the angles and sides of triangles may be accurately determined. But the calculations must be made by the tedious processes of multiplication and division. To avoid this inconvenience, another set of tables has been provided, in which are inserted the *logarithms* of the natural sines, tangents, &c. By the use of these, addition and subtraction are made to perform the office of multiplication and division. On this account, the tables of logarithmic, or as they are sometimes called, *artificial* sines, tangents, &c., are much more valuable, for practical purposes, than the *natural* sines, &c. Still it must be remembered that the former are derived from the latter. The artificial sine of an angle, is the logarithm of the natural sine of that angle. The artificial tangent is the logarithm of the natural tangent, &c.

103. One circumstance, however, is to be attended to, in comparing the two sets of tables. The radius to which the *natural* sines, &c., are calculated, is *unity.* (Art. 100.) The secants, and a part of the tangents are, therefore, *greater* than a unit; while the sines, and another part of the tangents, are *less* than a unit. When the logarithms of these are taken, some of the indices will be *positive,* and others *negative;* (Art. 9.) and the throwing of them together in the same table, if it does not lead to error, will at least be attended with inconvenience. To remedy this, 10 is added to each of the indices. (Art. 12.) They are then all positive. Thus the natural sine of 20° is 0.34202. The logarithm of this is $\bar{1}.53405$. But the index, by the addition of 10, be-

comes 10—1=9. The logarithmic sine in the tables is therefore 9.53405.*

Directions for taking Sines, Cosines, &c., from the tables.

104. The *cosine, cotangent,* and *cosecant* of an angle, are the sine, tangent, and secant of the *complement* of the angle. (Art. 89.) As the complement of an angle is the difference between the angle and 90°, and as 45 is the half of 90; if any given angle within the quadrant is greater than 45°, its complement is less; and, on the other hand, if the angle is *less* than 45°, its complement is greater. Hence, every cosine, cotangent, and cosecant of an angle greater than 45°, has its equal among the sines, tangents, and secants of angles less than 45°, and *v. v.*

Now, to bring the trigonometrical tables within a small compass, the same column is made to answer for the *sines* of a number of angles *above* 45°, and for the *cosines* of an equal number *below* 45°.

Thus 9.23967 is the log. *sine* of 10°, and the *cosine* of 80°,
9.53405 the *sine* of 20°, and the *cosine* of 70°, &c.

The tangents and secants are arranged in a similar manner. Hence,

105. *To find the Sine, Cosine, Tangent, &c., of any number of degrees and minutes.*

If the given angle is *less* than 45°, look for the degrees at the *top* of the table, and the minutes on the *left;* then, opposite to the minutes, and under the word sine at the head of the column, will be found the sine; under the word tangent, will be found the tangent, &c.

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* Or the tables may be supposed to be calculated to the radius 10000000000, whose logarithm is 10.

The log. sin	of 43° 25′ is	9.83715	The tan	of 17° 20′ is	9.49430
	of 17° 20′	9.47411		of 8° 46′	9.18812
The cos	of 17° 20′	9.97982	The cot	of 17° 20′	10.50570
	of 8° 46′	9.99490		of 8° 46′	10.81188

The first figure is the index; and the other figures are the decimal part of the logarithm.

106. If the given angle is between 45° and 90°; look for the degrees at the *bottom* of the table, and the minutes on the *right;* then, opposite to the minutes, and *over* the word sine at the foot of the column, will be found the sine; over the word tangent, will be found the tangent, &c.

Particular care must be taken, when the angle is less than 45°, to look for the title of the column, at the *top,* and for the minutes on the *left;* but when the angle is between 45° and 90°, to look for the title of the column at the *bottom,* and for the minutes, on the *right.*

The log. sine	of 81° 21′ is	9.99503
The cosine	of 72° 10′	9.48607
The tangent	of 54° 40′	10.14941
The cotangent	of 63° 22′	9.70026

107. If the given angle is *greater* than 90°, look for the sine, tangent, &c., of its *supplement.* (Art. 98, 99.)

The log. sine of	96° 44′ is	9.99699
The cosine of	171° 16′	9.99494
The tangent of	130° 26′	10.06952
The cotangent of	156° 22′	10.35894

108. *To find the sine, cosine, tangent, &c., of any number of degrees, minutes, and* SECONDS.

In the common tables, the sine, tangent, &c., are given only to every *minute* of a degree.* But they may be found to *seconds,* by taking *proportional parts* of the difference of the numbers as they stand in the tables. For, within a single minute, the variations in the sine, tangent, &c., are nearly proportional to the variations in the angle. Hence,

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* In the very valuable tables of Michael Taylor, the sines and tangents are given to *every second.*

To find the sine, tangent, &c., to seconds: Take out the number corresponding to the given degree and minute; and also that corresponding to the next greater minute, and find their difference. Then state this proportion;

As 60, to the given number of seconds;

So is the difference found, to the correction for the seconds.

This correction, in the case of sines, tangents, and secants, is to be *added* to the number answering to the given degree and minute; but for cosines, cotangents, and cosecants, the correction is to be *subtracted*;

For, as the sines *increase*, the cosines *decrease*.

Ex. 1. What is the logarithmic sine of 14° 43′ 10″?

The sine of	14° 43′ is	9.40490
of	14° 44′	9.40538
	Difference	48

Here it is evident that the sine of the required angle is greater than that of 14° 43′, but less than that of 14° 44′. And as the difference corresponding to a whole minute or 60″ is 48; the difference for 10″ must be a proportional part of 48. That is,

$$60'' : 10'' :: 48 : 8$$

the correction to be *added* to the sine of 14° 43′.

Therefore the sine of 14° 43′ 10″ is 9.40498.

2. What is the logarithmic cosine of 32° 16′ 45″?

The cosine of	32° 16′ is	9.92715
of	32° 17′	9.92707
	Difference	8

Then, $60'' : 45'' :: 8 : 6$ the correction to be *subtracted* from the cosine of 32° 16′.

Therefore the cosine of 32° 16′ 45″ is 9.92709.

The tangent of	24° 15′ 18″	is	9.65376
The cotangent of	31° 50′ 5″	is	10.20700
The sine of	58° 14′ 32″	is	9.92956
The cosine of	55° 10′ 26″	is	9.75670

If the given number of seconds be any even part of 60, as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c., the correction may be found, by taking a like part of the difference of the numbers in the tables, without stating a proportion in form.

109. *To find the degrees and minutes belonging to any given sine, tangent, &c.*

This is reversing the method of finding the sine, tangent, &c,. (Art. 105, 6, 7.)

Look in the column of the same name, for the sine, tangent, &c., which is *nearest* to the given one; and if the title be at the *head* of the column, take the degrees at the *top* of the table, and the minutes on the *left;* but if the title be at the *foot* of the column, take the degrees at the *bottom,* and the minutes on the *right.*

Ex. 1. What is the number of degrees and minutes belonging to the logarithmic sine 9.62863?

The nearest sine in the tables is 9.62865. The title of sine is at the head of the column in which these numbers are found. The degrees at the top of the page are 25, and the minutes on the left are 10. The angle required is, therefore 25° 10′.

The angle belonging to

the sine	9.87993	is 49° 20′	the cos	9.97351	is 19° 48′
the tan	9.97955	43° 39′	the cotan	9.75791	60° 12′
the sec	10.65396	77° 11′	the cosec	10.49066	18° 51′

110. *To find the degrees, minutes, and* SECONDS, *belonging to any given sine, tangent, &c.*

This is reversing the method of finding the sine, tangent, &c., to seconds. (Art. 108.)

First find the difference between the sine, tangent, &c., next greater than the given one, and that which is next less; then the difference between this less number and the given one; then

As the difference first found, is to the other difference;

So are 60 seconds, to the number of seconds, which, in the case of sines, tangents, and secants, are to be *added* to the degrees and minutes belonging to the least of the two numbers taken from the tables; but for cosines, cotangents, and cosecants are to be *subtracted.*

Ex. 1. What are the degrees, minutes, and seconds, belonging to the logarithmic sine 9.40498?

Sine next greater	14° 44′	9.40538	Given sine	9.40498
Next less	14° 43′	9.40490	Next less	9.40490
Difference		48	Difference	8

Then, 48 : 8 : : 60″ : 10″, which added to 14° 43′, gives 14° 43′ 10″ for the answer.

2. What is the angle belonging to the cosine 9.09773?

Cosine next greater	82° 48′	9.09807	Given cosine	9.09773
Next less	82° 49′	9.09707	Next less	9.09707
Difference		100	Difference	66

Then, 100 : 66 : : 60″ : 40″, which subtracted from 82° 49′, gives 82° 48′ 20″ for the answer.

It must be observed here, as in all other cases, that of the two angles, the less has the greater cosine.

The angle belonging to

the sin 9.20621 is	9° 15′ 6″	the tan 10.43434 is	69° 48′ 16″
the cos 9.98157	16° 34′ 30″	the cot 10.33554	24° 47′ 16″

Method of Supplying the Secants and Cosecants.

111. In some trigonometrical tables, the secants and cosecants are not inserted. But they may be easily obtained from the sines and cosines. For, by Art. 93, proportion 3d,

$$\cos \times \sec = R^2.$$

That is, the product of the cosine and secant, is equal to the square of radius. But, in logarithms, addition takes the place of multiplication; and, in the tables of logarithmic sines, tangents, &c., the radius is 10. (Art. 103.) Therefore, in these tables,

$$\cos + \sec = 20. \quad \text{Or } \sec = 20 - \cos.$$

Again, by Art 93, proportion 6,

$$\sin \times \operatorname{cosec} = R^2.$$

Therefore, in the tables,

$$\sin + \operatorname{cosec} = 20. \quad \text{Or, } \operatorname{cosec} = 20 - \sin. \quad \text{Hence,}$$

112. To obtain the *secant,* subtract the cosine from 20; and to obtain the *cosecant,* subtract the sine from 20.

These subtractions are most easily performed, by taking the right hand figure from 10, and the others from 9, as in finding the arithmetical complement of a logarithm; (Art. 55.) observing, however, to add 10 to the index of the secant or cosecant. In fact the secant is the arithmetical complement of the cosine, with 10 added to the index.

For the secant $= 20 - \cos.$
And the arith. comp. of cos $= 10 - \cos.$ (Art. 54.)

So also the cosecant is the arithmetical complement of the sine, with 10 added to the index. The tables of secants and cosecants are, therefore, of use, in furnishing the arithmetical complement of the sine and cosine, in the following simple manner:

113. For the arithmetical complement of the *sine,* subtract 10 from the index of the cosecant; and for the arithmetical complement of the *cosine,* subtract 10 from the index of the secant.

By this, we may save the trouble of taking each of the figures from 9.

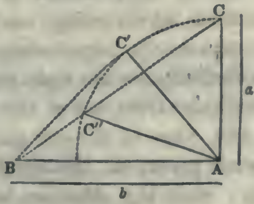
SECTION III.

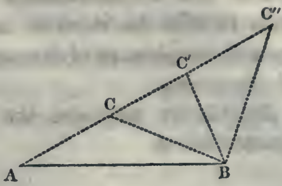
SOLUTIONS OF RIGHT ANGLED TRIANGLES.

ART. 114. In a triangle there are *six parts,* three sides, and three angles. In every trigonometrical calculation, it is necessary that some of these should be known, to enable us to find the others. *The number of parts which must be given, is* THREE, *one of which must be a* SIDE.

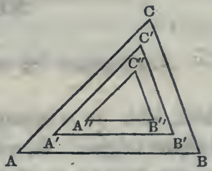
If only two parts be given, they will be either two sides, a side and an angle, or two angles; neither of which will limit the triangle to a particular form and size.

If *two sides* only be given, they may make any angle with each other; and may, therefore, be the sides of a thousand different triangles. Thus, the two lines *a* and *b* may belong either to the triangle ABC, or ABC′, or ABC″. So that it will be impossible, from knowing two of the sides of a triangle, to determine the other parts.





Or, if a *side and an angle* only be given, the triangle will be indeterminate. Thus, if the side AB and the angle at A be given; they may be parts either of the triangle ABC, or ABC′, or ABC″.



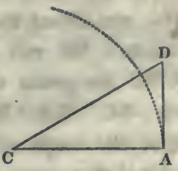
Lastly, if two *angles*, or even if *all* the angles be given, they will not determine the length of the sides. For the triangles ABC, A′B′C′, A″B″C″, and a hundred others which might be drawn, with sides parallel to these, will all have the same angles. So that one of the parts given must always be a side. If this and any other two parts, either sides or angles, be known, the other three may be found, as will be shown, in this and the following section.

115. Triangles are either *right angled* or *oblique angled.* The calculations of the former are the most simple, and those which we have the most frequent occasion to make. A great portion of the problems in the mensuration of heights and distances, in surveying, navigation and astronomy, are solved by rectangular trigonometry. Any triangle whatever may be divided into two right angled triangles, by drawing a perpendicular from one of the angles to the opposite side.

116. One of the six parts in a right angled triangle, is always given, viz. the right angle. This is a *constant* quantity; while the other angles and the sides are variable. It is also to be observed, that, if one of the *acute* angles is given, the other is known of course. For one is the complement of the other. (Art. 76, 77.) So that, *in a right angled triangle, subtracting one of the acute angles from* 90° *gives the*

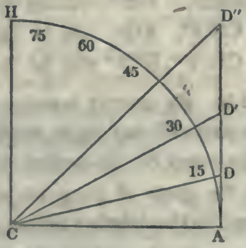
other. There remain, then, only *four* parts, one of the acute angles, and the three sides, to be sought by calculation. If any *two* of these be given, with the right angle, the others may be found.

117. To illustrate the method of calculation, let a case be supposed in which a right angled triangle CAD, has one of its sides equal to the radius to which the trigonometrical tables are adapted.



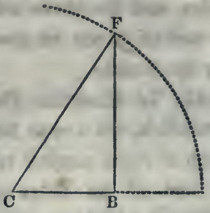
In the first place, let the *base* of the triangle be equal to the tabular radius. Then, if a circle be described, with this radius, about the angle C as a centre, DA will be the *tangent*, and DC the *secant* of that angle. (Art. 84, 85.) So that the radius, the tangent, and the secant of the angle at C, constitute the three sides of the triangle. The *tangent*, taken from the tables of natural sines, tangents, &c., will be the length of the *perpendicular*; and the *secant* will be the length of the *hypothenuse*. If the tables used be logarithmic, they will give the *logarithms* of the lengths of the two sides.

11



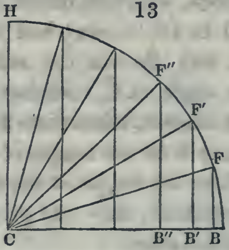
In the same manner, *any* right angled triangle whatever, whose base is equal to the radius of the tables, will have its other two sides found among the tangents and secants. Thus, if the quadrant AH, be divided into portions of 15° each; then, in the triangle

CAD, AD will be the tan, and CD the sec of 15°,
In CAD′, AD′ will be the tan, and CD′ the sec of 30°,
In CAD″, AD″ will be the tan, and CD″ the sec of 45°, &c.



118. In the next place, let the *hypothenuse* of a right angled triangle CBF, be equal to the radius of the tables. Then, if a circle be described, with the given radius, and about the angle C as a centre; BF will be the *sine,* and BC the *cosine* of that angle. (Art. 82, 89.) Therefore the sine of the angle at C, taken from the tables, will be the length of the *perpendicular,* and the cosine will be the length of the *base.*

13

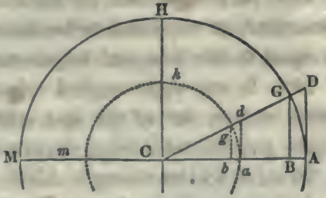


And any right angled triangle whatever, whose hypothenuse is equal to the tabular radius, will have its other two sides found among the sines and cosines. Thus, if the quadrant AH, be divided into portions of 15° each in the points F, F′, F″, &c.; then, in the triangle,

CBF, FB will be the sin, and CB the cos, of 15°,
In CB′F′, F′B′ will be the sin, and CB′ the cos, of 30°,
In CB″F″, F″B″ will be the sin, and CB″ the cos, of 45°, &c.

119. By merely *turning to the tables,* then, we may find the parts of any right angled triangle which has one of its sides equal to the radius of the tables. But for determining the parts of triangles which have *not* any of their sides equal to the tabular radius, the following proportion is used;

As the radius of one circle,
To the radius of any other;
So is a sine, tangent, or secant, in one,
To the sine, tangent, or secant, of the same number of degrees, in the other.



In the two concentric circles AHM, *ahm*, the arcs AG and *ag*, contain the same number of degrees. (Art. 74.) The sines of these arcs are BG and *bg*, the tangents AD and *ad*, and the secants CD and C*d*. The four triangles, CAD, CBG, C*ad*, and C*bg*, are similar. For each of them, from the nature of sines and tangents, contains one right angle; the angle at C is common to them all; and the other acute angle in each is the complement of that at C. (Art. 77.) We have, then, the following proportions. (Euc. 4. 6.*)

1. $CG : Cg :: BG : bg.$

That is, one radius is to the other, as one *sine* to the other.

2. $CA : Ca :: DA : da.$

That is, one radius is to the other, as one *tangent* to the other.

3. $CA : Ca :: CD : Cd.$

That is, one radius is to the other, as one *secant* to the other.

Cor. $BG : bg :: DA : da :: CD : Cd.$

That is, as the sine in one circle, to the sine in the other; so is the tangent in one, to the tangent in the other; and so is the secant in one, to the secant in the other.

This is a general principle, which may be applied to most trigonometrical calculations. If one of the sides of the proposed triangle be made radius, each of the other sides will be the sine, tangent, or secant, of an arc described by this radius. Proportions are then stated, between these lines, and the *tabular* radius, sine, tangent, &c.

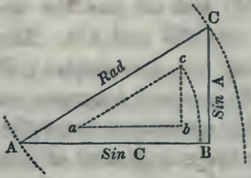
---

* Thomson 18. 4.

120. A line is said to be *made radius,* when a circle is described, or supposed to be described, whose semi-diameter is equal to the line, and whose centre is at one end of it.

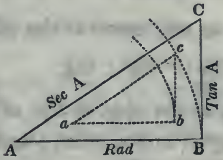
121. In any right angled triangle, *if the* HYPOTHENUSE *be made radius, one of the legs will be a* SINE *of its opposite angle, and the other leg a* COSINE *of the same angle.*

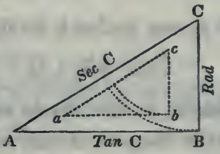
Thus, if to the triangle ABC a circle be applied whose radius is AC, and whose centre is A, then BC will be the *sine,* and BA the *cosine,* of the angle at A. (Art. 82, 89.)



If, while the same line is radius, the other end C be made the centre, then BA will be the *sine,* and BC the *cosine,* of the angle at C.

122. *If either of the* LEGS *be made radius, the other leg will be a* TANGENT *of its opposite angle, and the hypothenuse will be a* SECANT *of the same angle ;* that is, of the angle between the secant and the radius.





Thus, if the *base* AB (Fig. 15.) be made radius, the centre being at A, BC will be the *tangent,* and AC the *secant,* of the angle at A. (Art. 84. 85.)

But, if the *perpendicular* BC, (Fig. 16.) be made radius, with the centre at C, then AB will be the *tangent,* and AC the *secant,* of the angle at C.

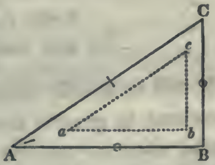
123. As the side which is the sine, tangent, or secant of one of the acute angles, is the cosine, cotangent, or cose-

cant of the other; (Art. 89.) the *perpendicular* BC (Fig. 14.) is the *sine* of the angle A, and the *cosine* of the angle C; while the *base* AB, is the *sine* of the angle C, and the *cosine* of the angle A.

If the base is made radius, as in Fig 15, the *perpendicular* BC is the *tangent* of the angle A, and the *cotangent* of the angle C; while the *hypothenuse* is the *secant* of the angle A, and the *cosecant* of the angle C.

If the perpendicular is made radius, as in Fig. 16, the *base* AB is the *tangent* of the angle C, and the *cotangent* of the angle A; while the *hypothenuse* is the *secant* of the angle C, and the *cosecant* of the angle A.

124. Whenever a right angled triangle is proposed, whose sides or angles are required; a *similar* triangle may be formed, from the sines, tangents, &c., of the *tables*. (Art. 117, 118.) The parts required are then found, by stating proportions between the similar sides of the two triangles. If the triangle proposed be ABC, (Fig. 17.) another *abc* may be formed, having the same angles with the first, but differing from it in the length of its sides, so as to correspond with the numbers in the tables. If similar sides be made radius in both, the remaining similar sides will be lines of *the same name;* that is, if the perpendicular in one of the triangles be a *sine,* the perpendicular in the other will be a sine; if the base in one be a *cosine,* the base in the other will be a cosine, &c.



If the *hypothenuse* in each triangle be made radius, as in Fig. 14, the perpendicular *bc*, will be the *tabular sine* of the angle at *a;* and the perpendicular BC, will be a sine of the equal angle A, in a circle of which AC is radius.

If the *base* in each triangle be made radius, as in Fig. 15, then the perpendicular *bc*, will be the *tabular tangent* of the angle at *a*; and BC will be a tangent of the equal angle A, in a circle of which AB, is radius, &c.

125. From the relations of the similar sides of these triangles, are derived the two following *theorems*, which are sufficient for calculating the parts of any right angled triangle whatever, when the requisite data are furnished. One is used, when a *side* is to be found; the other, when an *angle* is to be found.

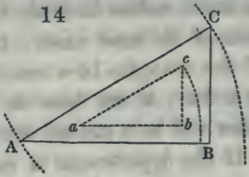
THEOREM I.

126. When a *side* is required;

> AS THE TABULAR SINE, TANGENT, &C., OF THE SAME NAME WITH THE GIVEN SIDE,
> TO THE GIVEN SIDE;
> SO IS THE TABULAR SINE, TANGENT, &C., OF THE SAME NAME WITH THE REQUIRED SIDE,
> TO THE REQUIRED SIDE.

It will be readily seen, that this is nothing more than a statement in general terms, of the proportions between the similar sides of two triangles, one proposed for solution, and the other formed from the numbers in the tables.

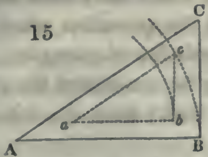
14



Thus, if the hypothenuse be *given*, and the base or perpendicular be *required*; then in Fig. 14, where *ac* is the tabular radius, *bc* the tabular sine of *a*, or its equal A, and *ab* the tabular sine of C; (Art. 124.)

$ac : AC :: bc : BC$, that is, $R : AC :: \sin A : BC$.
$ac : AC :: ab : AB$, $\qquad R : AC :: \sin C : AB$.

15



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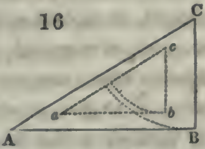


In Fig. 15, where ab is the tabular radius, ac the tabular secant of A, and bc the tabular tangent of A;

$ac : AC :: bc : BC$, that is, $\sec A : AC :: \tan A : BC$.
$ac : AC :: ab : AB$, $\sec A : AC :: R : AB$.

In Fig. 16, where bc is the tabular radius, ac the tabular secant of C, and ab the tabular tangent of C;

$ac : AC :: bc : BC$, that is, $\sec C : AC :: R : BC$.
$ac : AC :: ab : AB$, $\sec C : AC :: \tan C : AB$.

THEOREM II.

127. When an *angle* is required;

As the given side made radius,
To the tabular radius;
So is another given side,
To the tabular sine, tangent, &c., of the same name.

Thus, if the side made radius, and one other side be given, then, in Fig. 14,

$AC : ac :: BC : bc$, that is, $AC : R :: BC : \sin A$.
$AC : ac :: AB : ab$, $AC : R :: AB : \sin C$.

In Fig. 15,

$AB : ab :: BC : bc$, that is, $AB : R :: BC : \tan A$.
$AB : ab :: AC : ac$, $AB : R :: AC : \sec A$.

In Fig. 16,

$BC : bc :: AB : ab$, that is, $BC : R :: AB : \tan C$.
$BC : bc :: AC : ac$, $BC : R :: AC : \sec C$.

It will be observed that in these theorems, *angles* are not introduced, though they are among the quantities which are either given or required, in the calculation of triangles. But the tabular sines, tangents, &c., may be considered the *representatives* of angles, as one may be found from the other, by merely turning to the tables.

128. In the theorem for finding a *side*, the first term of the proportion is a *tabular number*. But, in the theorem for finding an *angle*, the first term is a side. Hence, in applying the proportions to particular cases, this rule is to be observed;

To find a SIDE, *begin with a tabular number,*
To find an ANGLE, *begin with a side.*

Radius is to be reckoned among the tabular numbers.

129. In the theorem for finding an *angle*, the first term is a *side made radius*. As in every proportion, the three first terms must be given to enable us to find the fourth, it is evident, that where this theorem is applied, the side made radius must be a *given* one. But, in the theorem for finding a *side*, it is not necessary that either of the terms should be radius. Hence,

130. *To find a* SIDE, ANY *side may be made radius,*
To find an ANGLE, *a* GIVEN *side must be made radius.*

It will generally be expedient, in both cases, to make radius one of the terms in the proportion; because, in the tables of natural sines, tangents, &c., radius is 1, and in the logarithmic tables it is 10. (Art. 103.)

131. The proportions in Trigonometry are of the same nature as other simple proportions. The fourth term is found, therefore, as in the Rule of Three in Arithmetic, by *multiplying together the second and third terms, and dividing their product by the first term.* This is the mode of calculation, when the tables of *natural* sines, tangents, &c., are used. But the operation by logarithms is so much more expeditious,

that it has almost entirely superseded the other method. In logarithmic calculations, addition takes the place of multiplication; and subtraction the place of division.

The logarithms expressing the lengths of the *sides* of a triangle, are to be taken from the tables of common logarithms. The logarithms of the *sines, tangents, &c.*, are found in the tables of artificial sines, &c. The calculation is then made *by adding the second and third terms, and subtracting the first.* (Art. 52.)

132. The logarithmic radius 10, or, as it is written in the tables, 10.00000, is so easily added and subtracted, that the three terms of which it is one, may be considered as, in effect, reduced to two. Thus, if the tabular radius is in the *first* term, we have only to add the other two terms, and then take 10 from the index; for this is subtracting the first term. If radius occurs in the *second* term, the first is to be subtracted from the third, after its index is increased by 10. In the same manner, if radius is in the *third* term, the first is to be subtracted from the second.

133. Every species of right angled triangles may be solved upon the principle, that the sides of similar triangles are proportional, according to the two theorems mentioned above. There will be some advantages, however, in giving the examples in distinct classes.

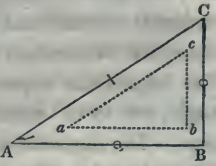
There must be given, in a right angled triangle, *two* of the parts, besides the right angle. (Art. 116.) These may be;

1. The hypothenuse and an angle; or
2. The hypothenuse and a leg; or
3. A leg and an angle; or
4. The two legs.

CASE I.

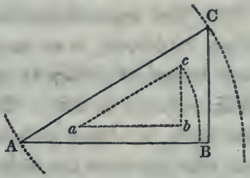
134. Given $\left\{\begin{array}{l}\text{The hypothenuse,}\\ \text{And an angle,}\end{array}\right\}$ to find $\left\{\begin{array}{l}\text{The base and}\\ \text{Perpendicular.}\end{array}\right.$

Ex. 1. If the hypothenuse AC,* be 45 miles, and the angle at A 32° 20′, what is the length of the base AB, and the perpendicular BC?



In this case, as *sides* only are required, *any* side may be made radius. (Art. 130.)

If the hypothenuse be made radius, BC will be the sine of A, and AB the sine of C, or the cosine of A. (Art. 121.) And if *abc* be a similar triangle, whose hypothenuse is equal to the *tabular* radius, *bc* will be the tabular sine of A, and *ab* the tabular sine of C. (Art. 124.)



To find the *perpendicular*, then, by Theorem I, we have this proportion;

$$ac : AC :: bc : BC.$$
$$\text{Or } R : AC :: \text{Sin } A : BC.$$

Whenever the terms Radius, Sine, Tangent, &c., occur in a proportion like this, the *tabular* Radius, &c., is to be understood, as in Arts. 126, 127.

The numerical calculation, to find the length of BC, may be made, either by *natural* sines, or by *logarithms.* See Art. 131.

By natural Sines.

$$1 : 45 :: 0.53484 : 24.068 = BC.$$

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* The parts which are *given* are distinguished by a mark across the line, or at the opening of the angle, and the parts *required* by a cipher.

Computation by Logarithms.

As radius		10.00000
To the hypothenuse	45	1.65321
So is the Sine of A	32° 20′	9.72823
To the perpendicular	24.068	1.38144

Here the logarithms of the second and third terms are added, and from the sum, the first term 10 is subtracted. (Art. 132.) The remainder is the logarithm of 24.068=BC.

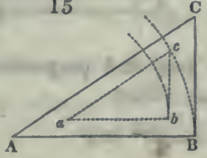
Subtracting the angle at A from 90°, we have the angle at C=57° 40′. (Art. 116.) Then to find the *base* AB;

$$ac : AC : : ab : AB$$

$$\text{Or } R : AC : : \text{Sin } C : AB = 38.023.$$

Both the sides required are now found, by making the hypothenuse radius. The results here obtained may be verified, by making either of the other sides radius.

15



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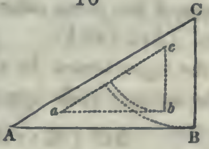


If the *base* be made radius, as in Fig. 15, the perpendicular will be the *tangent*, and the hypothenuse the *secant* of the angle at A. (Art. 122.) Then,

$$\text{Sec } A : AC : : R : AB$$

$$R : AB : : \text{Tan } A : BC$$

By making the arithmetical calculations, in these two proportions, the values of AB and BC, will be found the same as before.

If the *perpendicular* be made radius, as in Fig. 16, AB will be the *tangent*, and AC the *secant* of the angle at C. Then,

$$\text{Sec C} : \text{AC} :: \text{R} : \text{BC}$$
$$\text{R} : \text{BC} :: \text{Tan C} : \text{AB}$$

Ex. 2. If the hypothenuse of a right angled triangle be 250 rods, and the angle at the base 46° 30′; what is the length of the base and perpendicular?

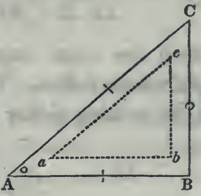
Ans. The base is 172.1 rods, and the perpendic. 181.35.

Case II.

135. Given $\left\{\begin{array}{l}\text{The hypothenuse,}\\ \text{And one leg.}\end{array}\right\}$ to find $\left\{\begin{array}{l}\text{The angles and}\\ \text{The other leg.}\end{array}\right.$

Ex. 1. If the hypothenuse be 35 leagues, and the base 26; what is the length of the perpendicular, and the quantity of each of the acute angles?

18



To find the angles it is necessary that one of the *given* sides be made radius. (Art. 130.)

If the *hypothenuse* be radius, the base and perpendicular will be sines of their opposite angles. Then,

$$\text{AC} : \text{R} :: \text{AB} : \text{Sin C} = 47° \ 58\tfrac{1}{2}'$$

And to find the *perpendicular* by Theorem I;

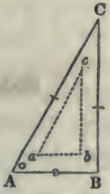
$$\text{R} : \text{AC} :: \text{Sin A} : \text{BC} = 23.43$$

If the *base* be radius, the perpendicular will be *tangent*, and the hypothenuse *secant* of the angle at A. Then,

$$\text{AB} : \text{R} :: \text{AC} : \text{Sec A}$$
$$\text{R} : \text{AB} :: \text{Tan A} : \text{BC}$$

In this example, where the hypothenuse and base are given, the angles cannot be found by making the *perpendicular* radius. For to find an angle, a *given* side must be made radius. (Art. 130.)

136. Ex. 2. If the hypothenuse be 54 miles, and the perpendicular 48 miles, what are the angles, and the base?



Making the *hypothenuse* radius.

AC : R : : BC : Sin A
R : AC : : Sin C : AB

The numerical calculation will give A=62° 44′ and AB =24.74.

Making the *perpendicular* radius.

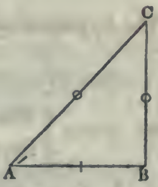
BC : R : : AC : Sec C
R : BC : : Tan C : AB

The angles cannot be found by making the *base* radius, when its length is not given.

Case III.

137. Given $\left\{\begin{array}{l}\text{The angles,}\\ \text{and one leg.}\end{array}\right\}$ to find $\left\{\begin{array}{l}\text{The hypothenuse,}\\ \text{and the other leg.}\end{array}\right.$

Ex. 1. If the base be 60, and the angle at the base 47° 12′, what is the length of the hypothenuse and the perpendicular?



In this case, as *sides* only are required, *any* side may be radius.

Making the *hypothenuse* radius.

Sin C : AB : : R : AC=88.31
R : AC : : Sin A : BC=64.8

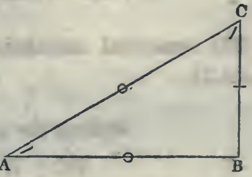
Making the *base* radius. (Fig. 20.)

R : AB : : Sec A : AC
R : AB : : Tan A : BC

Making the *perpendicular* radius.

Tan C : AB : : R : BC
R : BC : : Sec C : AC

138. Ex. 2. If the perpendicular be 74, and the angle C 61° 27′, what is the length of the base and the hypothenuse ?



Making the *hypothenuse* radius.

Sin A : BC : : R : AC
R : AC : : sin C : AB

Making the *base* radius.

Tan A : BC : : R : AB
R : AB : : sec A : AC

Making the *perpendicular* radius.

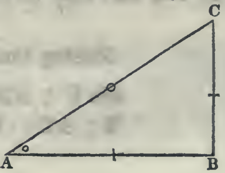
R : BC : : sec C : AC
R : BC : : tan C : AB

The hypothenuse is 154.83 and the base 136.

CASE IV.

$$139.\ \text{Given} \left\{ \begin{array}{l}\text{The base, and}\\ \text{Perpendicular,}\end{array}\right\} \text{to find} \left\{\begin{array}{l}\text{The hypothenuse,}\\ \text{And the angles.}\end{array}\right.$$

Ex. 1. If the base be 284, and the perpendicular 192, what are the angles, and the hypothenuse?



In this case, one of the legs

must be made radius, to find an angle; because the hypothenuse is not given.

Making the *base* radius.

$$AB : R :: BC : \tan A = 34° 4'$$
$$R : AB :: \sec A : AC = 342.84$$

Making the *perpendicular* radius.

$$BC : R :: AB : \tan C$$
$$R : BC :: \sec C : AC$$

Ex. 2. If the base be 640, and the perpendicular 480, what are the angles and hypothenuse?

Ans. The hypothenuse is 800, and the angle at the base 36° 52′ 12″.

Examples for Practice.

1. Given the hypothenuse 68, and the angle at the base 39° 17′; to find the base and perpendicular.
2. Given the hypothenuse 850, and the base 594, to find the angles, and the perpendicular.
3. Given the hypothenuse 78, and perpendicular 57, to find the base, and the angles.
4. Given the base 723, and the angle at the base 64° 18′, to find the hypothenuse and perpendicular.
5. Given the perpendicular 632, and the angle at the base 81° 36′, to find the hypothenuse and the base.
6. Given the base 32, and the perpendicular 24, to find the hypothenuse, and the angles.

140. The preceding solutions are all effected, by means of the tabular sines, tangents, and secants. But, when any *two sides* of a right angled triangle are given, the third side may be found, without the aid of the trigonometrical tables, by the proposition, that *the square of the hypothenuse is equal*

to the sum of the squares of the two perpendicular sides. (Euc. 47. 1.)

If the legs be given, extracting the square root of the *sum* of their squares, will give the hypothenuse. Or, if the hypothenuse and one leg be given, extracting the square root of the *difference* of the squares, will give the other leg.

$$\text{Let }\left.\begin{array}{l} h=\text{the hypothenuse} \\ p=\text{the perpendicular} \\ b=\text{the base} \end{array}\right\}\text{ of a right angled triangle.}$$

Then $h^2=b^2+p^2$, or (Alg. 248.) $h=\sqrt{b^2+p^2}$

By trans. $b^2=h^2-p^2$, or $b=\sqrt{h^2-p^2}$

And $p^2=h^2-b^2$, or $p=\sqrt{h^2-b^2}$

Ex. 1. If the base is 32, and the perpendicular 24, what is the hypothenuse? Ans. 40.

2. If the hypothenuse is 100, and the base 80, what is the perpendicular? Ans. 60.

3. If the hypothenuse is 300, and the perpendicular 220, what is the base?

Ans. $\overline{300}^2-\overline{220}^2=4160$, the root of which is 204 nearly.

141. It is generally most convenient to find the difference of the squares by *logarithms.* But this is not to be done by *subtraction.* For subtraction, in logarithms, performs the office of *division.* (Art. 41.) If we subtract the logarithm of b^2 from the logarithm of h^2, we shall have the logarithm, not of the *difference* of the squares, but of their *quotient.* There is, however, an indirect, though very simple method, by which the difference of the squares may be obtained by logarithms. It depends on the principle, that the *difference of the squares of two quantities is equal to the product of the sum and difference of the quantities.* (Alg. 191.) Thus,

$$h^2-b^2=(h+b)\times(h-b)$$

as will be seen at once, by performing the multiplication. The two factors may be multiplied by *adding* their logarithms. Hence,

142. *To obtain the difference of the squares of two quantities, add the logarithm of the sum of the quantities to the logarithm of their difference.* After the logarithm of the difference of the squares is found; the *square root* of this difference is obtained, by dividing the logarithm by 2. (Art. 47.)

Ex. 1. If the hypothenuse be 75 inches, and the base 45, what is the length of the perpendicular?

Sum of the given sides	120	log.	2.07918
Difference of do.	30		1.47712
	Dividing by		2)3.55630
Side required	60		1.77815

2. If the hypothenuse is 135, and the perpendicular 108, what is the length of the base? Ans. 81.

SECTION IV.

SOLUTIONS OF OBLIQUE ANGLED TRIANGLES.

Art. 143. The sides and angles of oblique angled triangles may be calculated by the following theorems.

Theorem I.

In any plane triangle, THE SINES OF THE ANGLES ARE AS THEIR OPPOSITE SIDES.

Let the angles be denoted by the letters A, B, C, and their opposite sides by *a*, *b*, *c*, as in Fig. 23 and 24. From one of the angles, let the line *p* be drawn perpendicular to the opposite side. This will fall either within or without the triangle.

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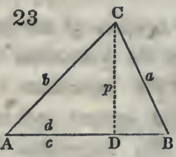

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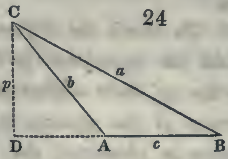

1. Let it fall *within* as in Fig. 23. Then, in the right angled triangles ACD, and BCD, according to Art. 126,

$$R : b :: \sin A : p$$
$$R : a :: \sin B : p$$

Here, the two *extremes* are the same in both proportions. The other four terms are, therefore, *reciprocally* proportional :[*] that is,

$$a : b :: \sin A : \sin B.$$

2. Let the perpendicular *p* fall *without* the triangle, as in Fig. 24. Then in the right angled triangles ACD and BCD;

$$R : b :: \sin A : p$$
$$R : a :: \sin B : p$$

Therefore, as before,

$$a : b :: \sin A : \sin B.$$

Sin A is here put both for the sine of DAC, and for that of BAC. For, as one of these angles is the *supplement* of the other, they have the same sine. (Art. 90.)

The sines which are mentioned here, and which are used in calculation are *tabular* sines. But the proportion will be the same, if the sines be adapted to any other radius. (Art. 119.)

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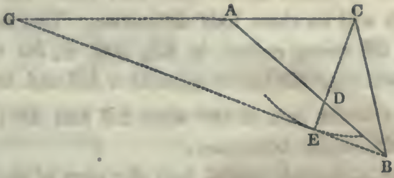
[*] Euclid, 23. 5.

Theorem II.

144. In a plane triangle,

As the sum of any two of the sides,
To their difference;
So is the tangent of half the sum of the opposite angles;
To the tangent of half their difference.



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Thus, the sum of AB and AC, is to their difference; as the tangent of half the sum of the angles ACB and ABC, to the tangent of half their difference.

Demonstration.

Extend CA to G, making AG equal to AB; then CG is *the sum of the two sides* AB and AC. On AB, set off AD, equal to AC; then BD *is the difference of the sides* AB and AC.

The sum of the two angles ACB and ABC, is equal to the sum of ACD and ADC; because each of these sums is the supplement of CAD. (Art. 79.) But as AC=AD by construction, the angle ADC=ACD (Euc. 5 1.*) Therefore ACD is *half the sum* of ACB and ABC. As AB=AG, the angle AGB=ABG, or DBE. Also, GCE, or ACD=ADC=BDE. (Euc. 15. 1.†) Therefore in the triangles

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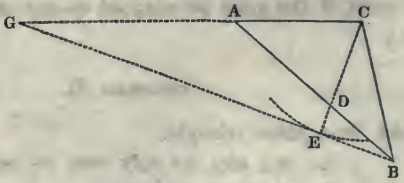
* Thomson's Legendre, 11. 1. † Ibid. 4. 1.



GCE, and DBE, the two remaining angles DEB, and CEG, are equal; (Art. 79.) So that CE is perpendicular to BG. (Euc. Def. 7. 1.*) If then CE is made radius, GE is the tangent of GCE, (Art. 84.) that is, *the tangent of half the sum of the angles opposite to* AB and AC.

If from the greater of the two angles ACB and ABC, there be taken ACD their half sum; the remaining angle ECB will be their half difference. The tangent of this angle, CE being radius, is EB, that is, *the tangent of half the difference of the angles opposite to* AB and AC. We have then,

CG=the sum of the sides AB and AC;
DB=their difference;
GE=the tangent of half the sum of the opposite angles;
EB=the tangent of half their difference.

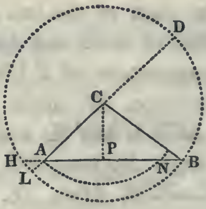
But by similar triangles,

CG : DB : : GE : EB. Q. E. D.

Theorem III.

145. If upon the longest side of a triangle, a perpendicular be drawn from the opposite angle;

As the longest side,
To the sum of the two others;
So is the difference of the latter,
To the difference of the segments made by the perpendicular.



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* Thomson's Legendre, Def. 12, 1.

In the triangle ABC, if a perpendicular be drawn from C upon AB;

$$AB : CB+CA :: CB-CA : BP-PA.^{*}$$

Demonstration.

Describe a circle on the centre C, and with the radius BC. Through A and C, draw the diameter LD, and extend BA to H. Then by (Euc. 35. 3.†)

$$AB\times AH=AL\times AD$$

Therefore,

$$AB : AD :: AL : AH$$

But $AD=CD+CA=CB+CA$
And $AL=CL-CA=CB-CA$
And $AH=HP-PA=BP-PA$ (Euc. 3. 3.—Thom. 6. 2.)

If, then, for the three last terms in the proportion, we substitute their equals, we have,

$$AB : CB+CA :: CB-CA : PB-PA.$$

146. It is to be observed, that the greater segment is next the greater side. If BC is greater than AC, PB is greater than AP. With the radius AC, describe the arc AN. The segment NP=AP. (Euc. 3. 3.) But BP is greater than NP.

147. The two segments are to each other, as the tangents of the opposite angles, or the cotangents of the adjacent angles. For, in the right angled triangles ACP, and BCP, if CP be made radius, (Art. 126.)

$$R : PC :: \text{Tan } ACP : AP$$
$$R : PC :: \text{Tan } BCP : BP$$

Therefore, by equality of ratios, (Alg. 346.‡)

$$\text{Tan } ACP : AP :: \text{Tan } BCP : BP$$

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* See note B. † Thomson's Legendre, 28. 4. Cor. ‡ Euc. 11. 5.

That is, the segments are as the tangents of the opposite angles. And the tangents of these are the *cotangents* of the adjacent angles A and B. (Art. 89.)

Cor. The greater segment is opposite to the greater angle. And of the angles at the base, the less is next the greater side. If BP is greater than AP, the angle BCP is greater than ACP ; and B is less than A. (Art. 77.)

148. To enable us to find the sides and angles of an oblique angled triangle, *three* of them must be *given*. (Art. 114.)

These may be, either

1. Two angles and a side, or
2. Two sides and an angle *opposite* one of them, or
3. Two sides and the *included* angle, or
4. The three sides.

The two first of these cases are solved by Theorem I, (Art 143.) the third by Theorem II, (Art. 144.) and the fourth by Theorem III. (Art. 145.)

149 In making the calculations, it must be kept in mind, that the greater side is always opposite to the greater angle, (Euc 18, 19. 1.*) that there can be only one *obtuse* angle in a triangle, (Art. 76.) and therefore, that the angles opposite to the two least sides must be *acute*.

Case I.

150. Given,

Two angles, and A side,	to find	The remaining angle, and The other two sides.

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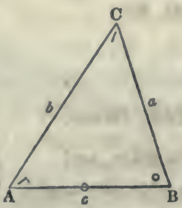
* Thomson's Legendre, 13. 1.

The third angle is found by merely subtracting the sum of the two which are given from 180°. (Art. 79.)

The sides are found, by stating, according to Theorem I, the following proportion;

As the sine of the angle opposite the *given* side,
To the length of the given side;
So is the sine of the angle opposite the *required* side
To the length of the required side.

As a *side* is to be found, it is necessary to begin with a *tabular number.*



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Ex. 1. In the triangle ABC, the side b is given 32 rods, the angle A 56° 20′, and the angle C 49° 10′, to find the angle B, and the sides a and c.

The sum of the two given angles $56° 20' + 49° 10' = 105° 30'$; which subtracted from 180°, leaves 74° 30′ the angle B. Then,

$$\text{Sin B} : b :: \left\{ \begin{array}{l} \text{Sin A} : a \\ \text{Sin C} : c \end{array} \right.$$

Calculation by logarithms.

As the sine of B	74° 30′	*a. c.*	0.01609
To the side b	32		1.50515
So is the sine of A	56° 20′		9.92027
To the side a	27.64		1.44151

As the sine of B	74° 30′	*a. c.*	0.01609
To the side b	32		1.50515
So is the sine of C	49° 10′		9.87887
To the side c	25.13		1.40011

The *arithmetical complement* used in the first term here, may be found in the usual way, or by taking out the *cosecant* of the given angle, and rejecting 10 from the index. (Art. 113.)

Ex. 2. Given the side b 71, the angle A 107° 6′, and the angle C 27° 40′; to find the angle B, and the sides a and c. The angle B is 45° 14′. Then,

$$\text{Sin B} : b :: \left\{ \begin{array}{l} \text{Sin A} : a = 95.58 \\ \text{Sin C} : c = 46.43 \end{array} \right.$$

When one of the given angles is *obtuse*, as in this example, the sine of its *supplement* is to be taken from the tables. (Art. 99.)

Case II.

151. Given,

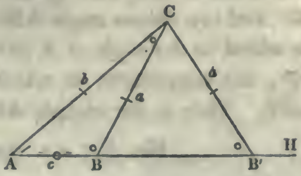
$$\left. \begin{array}{l} \text{Two sides, and} \\ \text{An opposite angle,} \end{array} \right\} \text{to find} \left\{ \begin{array}{l} \text{The remaining side and} \\ \text{The other two angles.} \end{array} \right.$$

One of the required angles is found, by beginning with a side, and, according to Theorem I, stating the proportion,

As the side opposite the given angle,
To the sine of that angle;
So is the side opposite the required angle,
To the sine of that angle.

The third angle is found, by subtracting the sum of the other two from 180°; and the remaining side is found, by the proportion in the preceding article.

152. In this second case, if the side opposite to the given angle be shorter than the other given side the solution will be *ambiguous*. Two different triangles may be formed, each of which will satisfy the conditions of the problem.



Let the side b, the angle A, and the length of the side opposite this angle be given. With the latter for radius, (if it be shorter than b,) describe an arc, cutting the line AH in the points B and B′. The lines BC and B′C, will be equal. So that, with the same data, there may be formed two different triangles, ABC and AB′C.

There will be the same ambiguity in the numerical calculation. The answer found by the proportion will be the *sine* of an angle. But this may be the sine either of the *acute* angle AB′C, or of the *obtuse* angle ABC. For, BC being equal to B′C, the angle CB′B is equal to CBB′. Therefore ABC, which is the supplement of CBB′, is also the supplement of CB′B. But the sine of an angle is the same, as the sine of its supplement. (Art. 90.) The result of the calculation will, therefore, be ambiguous. In practice, however, there will generally be some circumstances which will determine whether the angle required is acute or obtuse.

If the side opposite the given angle be *longer* than the other given side, the angle which is subtended by the latter, will necessarily be acute. For there can be but one obtuse angle in a triangle, and this is always subtended by the longest side. (Art. 149.)

If the *given* angle be obtuse, the other two will, of course, be acute. There can, therefore, be no ambiguity in the solution.

Ex. 1. Given the angle A, 35° 20′, the opposite side a 50, and the side b 70; to find the remaining side, and the other two angles.

To find the angle opposite to b, (Art. 151.)

$$a : \sin A :: b : \sin B$$

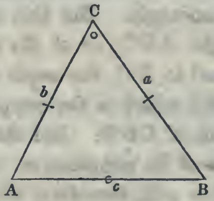
The calculation here gives the acute angle AB′C 54° 3′ 50″, and the obtuse angle ABC 125° 56′ 10″. If the latter be added to the angle at A 35° 20′, the sum will be 161° 16′ 10″, the supplement of which, 18° 43′ 50″, is the angle ACB. Then in the triangle ABC, to find the side c=AB,

$$\text{Sin A} : a :: \sin \text{C} : c = 27.76$$

If the *acute* angle AB′C 54° 3′ 50″ be added to the angle at A 35° 20′, the sum will be 89° 23′ 50″, the supplement of which, 90° 36′ 10″, is the angle ACB′. Then, in the triangle AB′C,

$$\text{Sin A} : \text{CB}' :: \sin \text{C} : \text{AB}' = 86.45.$$

Ex. 2. Given the angle at A, 63° 35′, the side b 64, and the side a 72; to find the side c, and the angles B and C.



$$a : \sin \text{A} :: b : \sin \text{B} = 52° 45' 25''$$
$$\text{Sin A} : a :: \sin \text{C} : c = 72.05$$

The sum of the angles A and B, is 116° 20′ 25″, the supplement of which, 63° 39′ 35″, is the angle C.

In this example the solution is *not ambiguous*, because the side opposite the given angle is longer than the other given side.

Ex. 3. In a triangle of which the angles are A, B, and C, and the opposite sides a, b, and c, as before; if the angle A be 121° 40′, the opposite side a 68 rods, and the side b 47 rods; what are the angles B and C, and what is the length of the side c? Ans. B is 36° 2′ 4″, C 22° 17′ 56″, and c 30.3.

In this example also, the solution is not ambiguous, because the *given* angle is obtuse.

CASE III.

153. Given,

Two sides, and The included angle,	to find	The remaining side, and The other two angles.

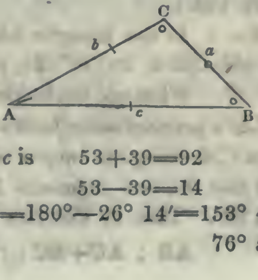
In this case, the angles are found by Theorem II. (Art. 144.) The required side may be found by Theorem I.

In making the solutions, it will be necessary to observe, that by subtracting the given angle from 180°, the *sum* of the other two angles is found; (Art. 79.) and, that *adding half the difference of two quantities to their half sum gives the greater quantity, and subtracting the half difference from the half sum gives the less.* The latter proposition may be geometrically demonstrated thus;

Let AE, be the greater of two magnitudes, and BE the less. Bisect AB in D, and make AC equal to BE. Then,

AB is the *sum* of the two magnitudes;
CE their *difference*;
DA or DB *half* their *sum*;
DE or DC *half* their *difference*;
But DA+DE=AE the *greater* magnitude;
And DE—DE=BE the *less*.

Ex. 1. In the triangle ABC, the angle A is given 26° 14′, the side *b* 39, and the side *c* 53; to find the angles B and C, and the side *a*.



The *sum* of the sides *b* and *c* is 53+39=92
And their *difference* 53—39=14
The sum of the angles B and C=180°—26° 14′=153° 46′
And *half the sum* of B and C is 76° 53′

Then, by Theorem II, (Fig. 30.)

$$(b+c) : (b-c) :: \tan \tfrac{1}{2}(B+C) : \tan \tfrac{1}{2}(B \sim C)$$

To and from the half sum	76°	53′	
Adding and subtracting the half difference	33	8	50
We have the greater angle	110	1	50
And the less angle	43	44	10

As the greater of the two given sides is c, the greater angle is C, and the less angle B. (Art. 149.)

To find the side a, by Theorem I.

$$\text{Sin } B : b :: \sin A : a = 24.94.$$

Ex. 2. Given the angle A 101° 30′, the side b 76, and the side c 109; to find the angles B and C, and the side a.

B is 30° 57½′, C 47° 32½′, and a 144.8

Case IV.

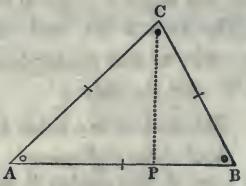
154. Given the three sides, to find the angles.

In this case, the solutions may be made, by drawing a perpendicular to the longest side, from the opposite angle. This will divide the given triangle into two *right angled* triangles. The two segments may be found by Theorem III. (Art. 145.)

There will then be given, in each of the right angled triangles, the hypothenuse and one of the legs, from which the angles may be determined, by rectangular trigonometry. (Art. 135.)

Ex. 1. In the triangle ABC, the side AB is 39, AC 35, and BC 27. What are the angles?

Let a perpendicular be drawn from C, dividing the longest side AB into the two segments AP and BP. Then by Theorem III,



$$AB : AC+BC :: AC-BC : AP-BP.$$

As the longest side	39 *a. c.*	8.40894
To the sum of the two others	62	1.79239
So is the difference of the latter	8	0.90309
To the difference of the segments	12.72	1.10442

The greater of the two segments is AP, because it is next the side AC, which is greater than BC. (Art. 146.)

To and from half the sum of the segments		19.5
Adding and subtracting half their difference, (Art. 153.)		6.36
We have the greater segment	AP	25.86
And the less	BP	13.14

Then, in each of the right angled triangles APC and BPC, we have given the hypothenuse and base; and by Art. 135.

$$AC : R :: AP : \cos A = 42^\circ\ 21'\ 57''$$
$$BC : R :: BP : \cos B = 60^\circ\ 52'\ 42''$$

And subtracting the sum of the angles A and B from 180°, we have the remaining angle ACB=76° 45′ 21″.

Ex. 2. If the three sides of a triangle are 78, 96, and 104; what are the angles?

Ans. 45° 41′ 48″, 61° 43′ 27″, and 72° 34′ 45″.

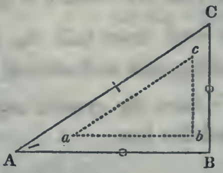
Examples for Practice.

1. Given the angle A 54° 30′, the angle B 63° 10′, and the side *a* 164 rods; to find the angle C, and the sides *b* and *c*.
2. Given the angle A 45° 6′, the opposite side *a* 93, and the side *b* 108; to find the angles B and C, and the side *c*.
3. Given the angle A 67° 24′, the opposite side *a* 62, and the side *b* 46; to find the angles B and C, and the side *c*.
4. Given the angle A 127° 42′, the opposite side *a* 381, and the side *b* 184; to find the angles B and C, and the side *c*.

5. Given the side *b* 58, the side *c* 67, and the included angle A=36°; to find the angles B and C, and the side *a*.
6. Given the three sides, 631, 268, and 546; to find the angles.

155. The three theorems demonstrated in this section, have been here applied to *oblique angled* triangles only. But they are equally applicable to *right angled* triangles.

Thus, in the triangle ABC, according to Theorem I, (Art. 143.)

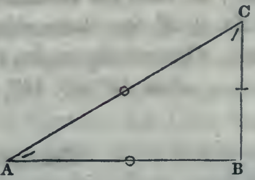


Sin B : AC : : sin A : BC

This is the same proportion as one stated in Art 134, except that, in the first term here, the *sine of* B is substituted for *radius*. But, as B is a right angle, its sine is *equal* to *radius*. (Art. 95.)

Again, in the triangle ABC, by the same theorem;

Sin A : BC : : sin C : AB



This is also one of the proportions in rectangular trigonometry, when the hypothenuse is made radius.

The other two theorems might be applied to the solution of right angled triangles. But, when one of the angles is *known* to be a right angle, the methods explained in the preceding section, are much more simple in practice.

SECTION V.

GEOMETRICAL CONSTRUCTION OF TRIANGLES, BY THE PLANE SCALE.

ART. 156. To facilitate the construction of geometrical figures, a number of graduated lines are put upon the common two feet scale; one side of which is called the *Plane Scale*, and the other side, *Gunter's Scale*. The most important of these are the scales of *equal parts*, and the line of *chords*. In forming a given triangle, or any other right lined figure, the parts which must be made to agrée with the conditions proposed, are the *lines* and the *angles*. For the former, a scale of equal parts is used; for the latter, a line of chords.

157. The line on the upper side of the plane scale, is divided into *inches* and *tenths* of an inch. Beneath this, on the left hand, are two *diagonal* scales of equal parts,* divided into inches and half inches, by perpendicular lines. On the larger scale, one of the inches is divided into tenths, by lines which pass *obliquely* across, so as to intersect the parallel lines which run from right to left. The use of the oblique lines is to measure *hundredths* of an inch, by inclining more and more to the right, as they cross each of the parallels.

To take off, for instance, an extent of 3 inches, 4 tenths, and 6 hundredths;

Place one foot of the dividers at the intersection of the perpendicular line marked 3 with the parallel line marked 6, and the other foot at the intersection of the latter with the oblique line marked 4.

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* These lines are not represented by a figure, as the learner is supposed to have the scale before him.

The other diagonal scale is of the same nature. The divisions are smaller, and are numbered from left to right.

158. In geometrical constructions, what is often required, is to make a figure, not *equal* to a given one, but only *similar.* Now figures are similar which have equal angles, and the sides about the equal angles *proportional.* (Euc. Def. 1. 6.*) Thus a land surveyor, in plotting a field, makes the several lines in his plan to have the same proportion to each other, as the sides of the field. For this purpose a scale of equal parts may be used, of any dimensions whatever. If the sides of the field are 2, 5, 7, and 10 *rods,* and the lines in the plan are 2, 5, 7, and 10 *inches,* and if the angles are the same in each, the figures are similar. One is a copy of the other, upon a smaller scale.

So any two right lined figures are similar, if the angles are the same in both, and if the number of smaller parts in each side of one, is equal to the number of larger parts in the corresponding sides of the other. The several divisions on the scale of equal parts may, therefore, be considered as representing any measures of length, as feet, rods, miles, &c. All that is necessary is, that the scale be not changed, in the construction of the same figure; and that the several divisions and subdivisions be properly proportioned to each other. If the larger divisions, on the diagonal scale, are units, the smaller ones are tenths and *hundredths.* If the larger are tens, the smaller are units and tenths.

159. In laying down an *angle,* of a given number of degrees, it is necessary to *measure* it. Now the proper measure of an angle is an arc of a circle. (Art. 74.) And the measure of an arc, where the radius is given, is its *chord.* For the chord is the distance, in a straight line, from one

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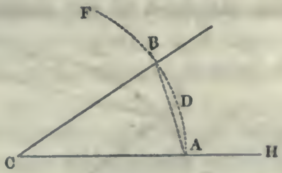
* Thomson's Legendre, Def. 3. 4.



end of the arc to the other. Thus the chord AB, is a measure of the arc ADB, and of the angle ACB.

To form the *line of chords*, a circle is described, and the length of its chords determined for every degree of the quadrant. These measures are put on the plane scale, on the line marked CHO.

160. The chord of 60° is equal to *radius*. (Art. 95.) In laying down or measuring an angle, therefore, an arc must be drawn, with a radius which is equal to the extent from 0 to 60 on the line of chords. There are generally on the scale, two lines of chords. Either of these may be used; but the angle must be measured by the same line from which the radius is taken.

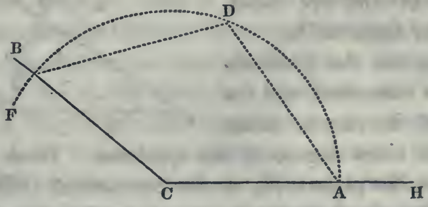
161. To *make an angle*, then, of a given number of degrees; from one end of a straight line as a centre, and with a radius equal to the chord of 60° on the line of chords, describe an arc of a circle cutting a straight line. From the point of intersection, extend the chord of the given number of degrees, applying the other extremity to the arc; and through the place of meeting, draw the other line from the angular point.

If the given angle is *obtuse*, take from the scale the chord of *half* the number of degrees, and apply it *twice* to the arc. Or make use of the chords of any two arcs whose *sum* is equal to the given number of degrees.

A *right angle* may be constructed, by drawing a perpendicular without using the line of chords.

Ex. 1. To make an angle of 32 degrees. With the point C, in the line CH, for a centre, and with the chord of 60° for radius, describe the arc ADF. Extend the chord of 32° from A to B; and through B, draw the line BC. Then is ACB an angle of 32 degrees.

2. To make an angle of 140 degrees. On the line CH, with the chord of 60°, describe the arc ADF; and extend the chord of 70° from A to D, and from D to B. The arc ADB=70°×2=140°.



On the other hand:

162. To *measure an angle;* On the angular point as a centre, and with the chord of 60° for radius, describe an arc to cut the two lines which include the angle, The distance between the points of intersection, applied to the line of chords, will give the measure of the angle in degrees. If the angle be *obtuse,* divide the arc into two parts.

Ex. 1. To measure the angle ACB. (Fig. 33, page 101.) Describe the arc ADF, cutting the lines CH and CB. The distance AB, will extend 32° on the line of chords.

2. To measure the angle ACB. (Fig. 34.) Divide the arc ADB into two parts, either equal or unequal, and measure each part, by applying its chord to the scale. The sum of the two will be 140°.

163. Besides the lines of chords, and of equal parts, on the plane scale; there are also lines of natural *sines, tangents,* and *secants,* marked Sin., Tan., and Sec.; of *semitangents,* marked S. T.; of *longitude,* marked Lon. or M. L.; of *rhumbs,* marked Rhu. or Rum., &c. These are not necessary in trigonometrical construction. Some of them are used in Navigation; and some of them in the projections of the Sphere.

164. In Navigation, the quadrant, instead of being graduated in the usual manner, is divided into *eight* portions, called *Rhumbs.* The *Rhumb line,* on the scale, is a line of chords, divided into rhumbs and quarter-rhumbs, instead of degrees.

165. The line of *Longitude* is intended to show the number of geographical miles in a degree of longitude, at different distances from the equator. It is placed over the line of chords, with the numbers in an inverted order: so that the figure above shows the length of a degree of longitude, in any latitude denoted by the figure below.* Thus, at the equator, where the latitude is 0, a degree of longitude is 60 geographical miles. In latitude 40, it is 46 miles; in latitude 60, 30 miles, &c.

166. The graduation on the line of *secants* begins where the line of sines ends. For the greater sine is only equal to radius; but the secant of the least arc is greater than radius.

167. The *semitangents* are the tangents of *half* the given arcs. Thus, the semitangent of 20° is the tangent of 10°. The line of semitangents is used in one of the projections of the sphere.

168. In the construction of *triangles,* the sides and angles which are *given,* are laid down according to the directions in Arts. 158, 161. The parts *required* are then measured, according to Arts. 158, 162. The following problems correspond with the four cases of oblique angled triangles; (Art. 148.) but are equally adapted to right angled triangles.

169. PROB. I. *The angles and one side* of a triangle being given; to find, by construction, the other two sides.

Draw the given side. From the ends of it, lay off two of the given angles. Extend the other sides till they intersect; and then measure their lengths on a scale of equal parts.

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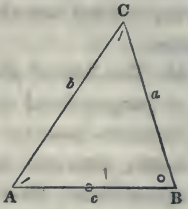
* Sometimes the line of longitude is placed *under* the line of chords.

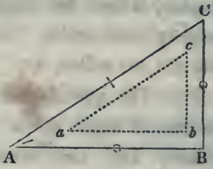
Ex. 1. Given the side *b* 32 rods, the angle A 56° 20′, and the angle C 49° 10′; to construct the triangle, and find the lengths of the sides *a* and *c*.

Their lengths will be 25 and 27½.



2. In a right angled triangle, given the hypothenuse 90, and the angle A 32° 20′, to find the base and perpendicular.

The length of AB will be 76, and of BC 48.



3. Given the side AC 68, the angle A 124°, and the angle C 37°: to construct the triangle.

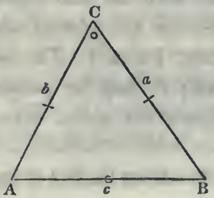
170. Prob. II. *Two sides and an opposite angle* being given, to find the remaining side, and the other two angles.

Draw one of the given sides; from one end of it, lay off the given angle; and extend a line indefinitely for the required side. From the other end of the first side, with the remaining given side for radius, describe an arc cutting the indefinite line. The point of intersection will be the end of the required side.

If the side opposite the given angle be less than the other given side, the case will be *ambiguous.* (Art. 152.)

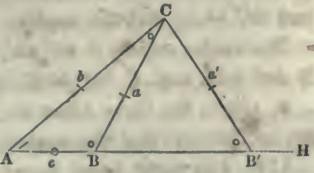
Ex. 1. Given the angle A 63° 35′, the side *b* 32, and the side *a* 36.

The side AB will be 36 nearly, the angle B 52° 45½′, and C 63° 39½′.



2. Given the angle A 35° 20′, the opposite side *a* 25, and the side *b* 35.

Draw the side *b* 35, make the angle A 35° 20′, and extend AH indefinitely. From C with radius 25, describe an arc cutting AH in B and B′. Draw CB and CB′, and two triangles will be formed, ABC and AB′C, each corresponding with the conditions of the problem.



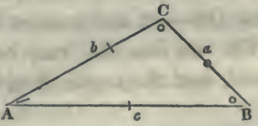
3. Given the angle A 116°, the opposite side *a* 38, and the side *b* 26; to construct the triangle.

171. Prob. III. *Two sides and the included angle* being given; to find the other side and angles.

Draw one of the given sides. From one end of it lay off the given angle, and draw the other given side. Then connect the extremities of this and the first line.

Ex. 1. Given the angle A 26° 14′, the side *b* 78, and the side *c* 106; to find B, C, and *a*.

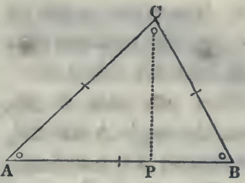
The side *a* will be 50, the angle B 43° 44′, and C 110° 2′. 30



2. Given A 86°, *b* 65, and *c* 83; to find B, C, and *a*.

172. Prob. IV. The *three sides* being given; to find the angles.

Draw one of the sides, and from one end of it, with an extent equal to the second side, describe an arc. From the other end, with an extent equal to the third side, describe a second arc cutting the first; and from the point of intersection draw the two sides. (Euc. 22. 1.)

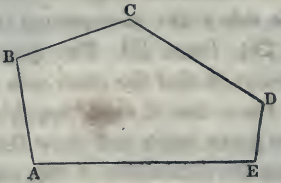


Ex. 1. Given AB 78, AC 70, and BC 54, to find the angles.

The angles will be A 42° 22′, B 60° $52\frac{2}{3}'$ and C 76° $45\frac{1}{3}'$.

2. Given the three sides 58, 39, and 46; to find the angles.

173. Any right lined figure whatever, whose sides and angles are given, may be constructed, by laying down the sides from a scale of equal parts, and the angles from a line of chords.



Ex. Given the sides AB=20, BC=22, CD=30, DE=12; and the angles B=102°, C=130°, D=108°, to construct the figure.

Draw the side AB=20, make the angle B=102°, draw BC=22, make C=130°, draw CD=30, make D=108°, draw DE=12, and connect E and A.

The last line, EA, may be measured on the scale of equal parts; and the angles E and A, by a line of chords.

SECTION VI.

DESCRIPTION AND USE OF GUNTER'S SCALE.

Art. 174. An expeditious method of solving the problems in trigonometry, and making other logarithmic calculations, in a mechanical way, has been contrived by Mr. Edmund Gunter. The logarithms of numbers, of sines, tangents, &c., are represented by *lines*. By means of these, multiplication, division, the rule of three, involution, evolution, &c., may be performed much more rapidly, than in the usual method by figures.

The logarithmic lines are generally placed on one side only of the scale in common use. They are,

A line of artificial *Sines* divided into *Rhumbs*, and marked,		S. R.
A line of artificial *Tangents*,	do	T. R.
A line of the logarithms of *Numbers*,		Num.
A line of artificial *Sines*, to every *degree*,		SIN.
A line of artificial *Tangents*,	do	TAN.
A line of *Versed Sines*.		V. S.

To these are added a line of *equal parts*, and a line of *Meridional Parts*, which are not logarithmic. The latter is used in Navigation.

The Line of Numbers.

175. Portions of the line of *Numbers*, are intended to represent the *logarithms* of the natural series of numbers 2, 3, 4, 5, &c.

The logarithms of 10, 100, 1000, &c., are 1, 2, 3, &c. (Art. 3.)

If, then, the log. of 10 be represented by a line of 1 foot;
the log. of 100 will be repres'd by one of 2 feet;
the log. of 1000 by one of 3 feet;
the lengths of the several lines being proportional to the corresponding logarithms in the tables. *Portions* of a foot will represent the logarithms of numbers between 1 and 10; and portions of a line 2 feet long, the logarithms of numbers between 1 and 100.

On Gunter's scale, the line of the logarithms of numbers begins at a brass pin on the left, and the divisions are numbered 1, 2, 3, &c., to another pin near the middle. From this the numbers are repeated, 2, 3, 4, &c., which may be read 20, 30, 40, &c. The logarithms of numbers between 1 and 10, are represented by portions of the first half of the line; and the logarithms of numbers between 10 and 100, by portions greater than half the line, and less than the whole.

176. The logarithm of 1, which is 0, is denoted, not by any extent of line, but by a *point* under 1, at the commencement of the scale. The distances from this point to different parts of the line, represent other logarithms, of which the *figures* placed over the several divisions are the *natural numbers*. For the intervening logarithms, the intervals between the figures, are divided into tenths, and sometimes into smaller portions. On the right hand half of the scale, as the divisions which are numbered are *tens*, the subdivisions are units.

Ex. 1. To take from the scale the logarithm of 3.6; set one foot of the dividers under 1 at the beginning of the scale, and extend the other to the 6th division after the first figure 3.

2. For the logarithm of 47; extend from 1 at the beginning, to the 7th subdivision after the second figure 4*.

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* If the dividers will not reach the distance required; first open them so as to take off *half*, or any part of the distance, and then the remaining part.

177. It will be observed, that the divisions and subdivisions *decrease*, from left to right; as in the tables of *logarithms*, the differences decrease. The difference between the logarithms of 10 and 100, is no greater, than the difference between the logarithms of 1 and 10.

178. The line of numbers, as it has been here explained, furnishes the logarithms of all numbers between 1 and 100.

And if the indices of the logarithms be neglected, the same scale may answer for all numbers whatever. For the *decimal* part of the logarithm of any number is the same, as that of the number multiplied or divided by 10, 100, &c. (Art. 14.) In logarithmic calculations, the use of the indices is to determine the distance of the several figures of the natural numbers from the place of units. (Art. 11.) But in those cases in which the logarithmic line is commonly used, it will not generally be difficult to determine the local value of the figures in the result.

179. We may, therefore, consider the *point* under 1 at the left hand, as representing the logarithm of 1, or 10, or 100; or $\frac{1}{10}$, or $\frac{1}{100}$, &c., for the decimal part of the logarithm of each of these is 0. But if the first 1 is reckoned 10, all the succeeding numbers must also be increased in a tenfold ratio; so as to read, on the first half of the line, 20, 30, 40, &c., and on the other half, 200, 300, &c.

The whole extent of the logarithmic line,

is from 1 to 100, or from 0.1 to 10,
or from 10 to 1000, or from 0.01 to 1,
or from 100 to 10000, &c. or from 0.001 to 0.1, &c.

Different values may, on different occasions, be assigned to the several numbers and subdivisions marked on this line. But for any one calculation, the value must remain the same.

Ex. Take from the scale 365.

As this number is between 10 and 1000, let the 1 at the beginning of the scale, be reckoned 10. Then, from this point to the second 3 is 300; to the 6th dividing stroke is 60; and half way from this to the next stroke is 5.

180. Multiplication, division, &c., are performed by the line of numbers, on the same principle, as by common logarithms. Thus,

To *multiply* by this line, *add* the logarithms of the two factors; (Art. 37.) that is, take off, with the dividers, that length of line which represents the logarithm of *one* of the factors, and apply this so as to extend forward from the end of that which represents the logarithm of the *other* factor. The sum of the two will reach to the end of the line representing the logarithm of the product.

Ex. Multiply 9 into 8. The extent from 1 to 8, added to that from 1 to 9, will be equal to the extent from 1 to 72, the product.

181. To *divide* by the logarithmic line, *subtract* the logarithm of the divisor from that of the dividend; (Art. 41.) that is, take off the logarithm of the divisor, and this extent set back from the end of the logarithm of the dividend, will reach to the logarithm of the quotient.

Ex. Divide 42 by 7. The extent from 1 to 7, set back from 42, will reach to 6, the quotient.

182. *Involution* is performed in logarithms, by multiplying the logarithm of the quantity into the index of the power; (Art. 45.) that is, by *repeating* the logarithms as many times as there are units in the index. To involve a quantity on the scale, then, take in the dividers the linear logarithm, and *double it, treble it, &c.*, according to the index of the proposed power.

Ex. 1. Required the square of 9. Extend the dividers from 1 to 9. *Twice* this extent will reach to 81, the square.

2. Required the cube of 4. The extent from 1 to 4 repeated *three times*, will reach to 64 the cube of 4.

183. On the other hand, to perform *evolution* on the scale;

take *half, one-third, &c.*, of the logarithm of the quantity, according to the index of the proposed root.

Ex. 1. Required the square root of 49. *Half* the extent from 1 to 49, will reach from 1 to 7, the root.

2. Required the cube root of 27. *One third* the distance from 1 to 27, will extend from 1 to 3, the root.

184. The *Rule of Three* may be performed on the scale, in the same manner as in logarithms, by adding the two middle terms, and from the sum, subtracting the first term (Art. 52.) But it is more convenient in practice to *begin* by subtracting the first term from one of the others. If four quantities are proportional, the quotient of the first divided by the second, is equal to the quotient of the third divided by the fourth. (Alg. 315.)

Thus, if $a : b :: c : d$, then $\frac{a}{b}=\frac{c}{d}$, and $\frac{a}{c}=\frac{b}{d}$. (Alg. 344.)

But in logarithms, *subtraction* takes the place of division; so that,

$\log. a - \log. b = \log. c - \log. d$. Or, $\log. a - \log. c = \log. b - \log. d$.

Hence,

185. On the scale, *the difference between the first and second terms of a proportion, is equal to the difference between the third and fourth.* Or, the difference between the first and third terms, is equal to the difference between the second and fourth.

The difference between the two terms is taken, by extending the dividers from one to the other. If the second term be greater than the first; the fourth must be greater than the third; if less, less.* Therefore, if the dividers extend *forward* from *left to right*, that is, from a less number to a greater, from the first term to the second;

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* Euclid, 14, 5.

they must also extend forward from the third to the fourth. But if they extend *backward,* from the first term to the second; they must extend the same way, from the third to the fourth.

Ex. 1. In the proportion 3 : 8 : : 12 : 32, the extent from 3 to 8, will reach from 12 to 32; Or, the extent from 3 to 12, will reach from 8 to 32.

2. If 54 yards of cloth cost 48 dollars, what will 18 yds. cost?

54 : 48 : : 18 : 16

The extent from 54 to 48, will reach *backwards* from 18 to 16.

3. If 63 gallons of wine cost 81 dollars, what will 35 gallons cost?

63 : 81 : : 35 : 45

The extent from 63 to 81, will reach from 35 to 45.

The Line of Sines.

186. The line on Gunter's scale marked SIN. is a line of logarithmic sines, made to correspond with the line of numbers. The whole extent of the line of numbers, (Art, 179.)

is from 1 to 100, whose logs. are 0.00000 and 2.00000,
or from 10 to 1000, whose logs. are 1.00000 and 3.00000,
or from 100 to 10000, whose logs. are 2.00000 4.00000,

the *difference of the indices* of the two extreme logarithms being in each case 2.

Now the logarithmic sine of 0° 34′ 22″ 41‴ is 8.00000
And the sine of 90° (Art. 95.) is 10.00000

Here also the difference of the indices is 2. If then the point directly beneath one extremity of the line of numbers, be marked for the sine of 0° 34′ 22″ 41‴; and the point beneath the other extremity, for the sine of 90° ; the interval may furnish the intermediate sine ; the divisions on it being made to correspond with the decimal part of the logarithmic sines in the tables.*

The first dividing stroke in the line of Sines is generally at 0° 40′, a little farther to the right than the beginning of the line of numbers. The next division is at 0° 50′ ; then begins the numbering of the degrees, 1, 2, 3, 4, &c., from left to right.

The Line of Tangents.

187. The first 45 degrees on this line are numbered from left to right, nearly in the same manner as on the line of Sines.

The logarithmic tangent of 0° 34′ 22″ 35‴ is 8.00000
And the tangent of 45°, (Art. 95.) is 10.00000

The difference of the indices being 2, 45 degrees will reach to the end of the line. For those above 45° the scale ought to be continued much farther to the right. But as this would be inconvenient, the numbering of the degrees, after reaching 45, is *carried back* from right to left. The same dividing stroke answers for an arc and its *complement*, one above and the other below 45°. For, (Art. 93. Propor. 9.)

$$\tan : R :: R : \cot.$$

In logarithms, therefore, (Art. 184.)

$$\tan - R = R - \cot.$$

That is, the *difference* between the tangent and radius, is equal to the difference between radius and the cotangent : in

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[*] To represent the sines *less* than 34′ 22″ 41‴, the scale must be extended on the left indefinitely. For, as the sine of an arc approaches to 0, its logarithm, which is negative, increases without limit. (Art. 15.)

other words, one is as much *greater* than the tangent of 45°, as the other is *less.* In taking, then, the tangent of an arc greater than 45°, we are to suppose the distance between 45 and the division marked with a given number of degrees, to be added to the whole line, in the same manner as if the line were continued out. In working proportions, extending the dividers *back,* from a less number to a greater, must be considered the same as carrying them *forward* in other cases. See Art. 185.

Trigonometrical Proportions on the Scale.

188. In working proportions in trigonometry by the scale; *the extent from the first term to the middle term of the same name, will reach from the other middle term to the fourth term.* (Art. 185.)

In a trigonometrical proportion, two of the terms are the lengths of sides of the given triangle; and the other two are tabular sines, tangents, &c. The former are to be taken from the line of numbers; the latter, from the lines of logarithmic sines and tangents. If one of the terms is a *secant,* the calculation cannot be made on the scale, which has commonly no line of secants. It must be kept in mind that *radius* is equal to the sine of 90°, or to the tangent of 45°. (Art. 95.) Therefore, whenever radius is a term in the proportion, one foot of the dividers must be set on the end of the line of sines or of tangents.

189. The following examples are taken from the proportions which have already been solved by numerical calculation.

Ex. 1. In Case I, of right angled triangles, (Art. 134. ex. 1.)

$$R : 45 :: \sin 32^\circ\, 20' : 24$$

Here the third term is a *sine;* the first term radius is, therefore, to be considered as the sine of 90°. Then the extent from 90° to 32° 20′ on the line of sines, will reach from 45 to 24 on the line of numbers. As the dividers are set *back* from 90° to 32° 20′; they must also be set back from 45. (Art. 185.)

2. In the same case, if the base be made radius, (page 60.)

R : 38 : : tan 32° 20′ : 24

Here, as the third term is a *tangent,* the first term radius is to be considered the tangent of 45°. Then the extent from 45° to 32° 20′ on the line of tangents, will reach from 38 to 24 on the line of numbers.

3. If the perpendicular be made radius, (page 62.)

R : 24 : : tan 57° 40′ : 38

The extent from 45° to 57° 40′ on the line of tangents, will reach from 24 to 38 on the line of numbers. For the tangent of 57° 40′ on the scale, look for its *complement* 32° 20′. (Art. 187.) In this example, although the dividers extend *back* from 45° to 57° 40′; yet, as this is from a *less* number to a *greater,* they must extend *forward* on the line of numbers. (Arts. 185, 187.)

4. In Art. 135,

35 : R : : 26 : sin 48°

The extent from 35 to 26 will reach from 90° to 48°.

5. In Art. 136,

R : 48 : : tan $27\frac{1}{4}$° : $24\frac{3}{4}$

The extent from 45° to $27\frac{1}{4}$°, will reach from 48 to $24\frac{3}{4}$.

6. In Art. 150, ex. 1.

Sin 74° 30′ : 32 : : sin 56° 20′ : $27\frac{1}{2}$.

For other examples, see the several cases in Sections III. and IV.

190. Though the solutions in trigonometry may be ef-

fected by the logarithmic scale, or by geometrical construction, as well as by arithmetical computation; yet the latter method is by far the most accurate. The first is valuable principally for the *expedition* with which the calculations are made by it. The second is of use, in presenting the *form* of the triangle to the eye. But the accuracy which attends arithmetical operations, is not to be expected, in taking lines from a scale with a pair of dividers.*

SECTION VII.

THE FIRST PRINCIPLES OF TRIGONOMETRICAL ANALYSIS.

ART. 191. In the preceding sections, sines, tangents, and secants have been employed in calculating the sides and angles of triangles. But the use of these lines is not confined to this object. Important assistance is derived from them, in conducting many of the investigations in the higher branches of analysis, particularly in physical astronomy. It does not belong to an elementary treatise of trigonometry, to prosecute these inquiries to any considerable extent. But this is the proper place for *preparing the formulæ*, the applications of which are to be made elsewhere.

Positive and Negative SIGNS *in Trigonometry.*

192. Before entering on a particular consideration of the algebraic expressions which are produced by combinations of the several trigonometrical lines, it will be necessary to attend to the positive and negative *signs* in the different

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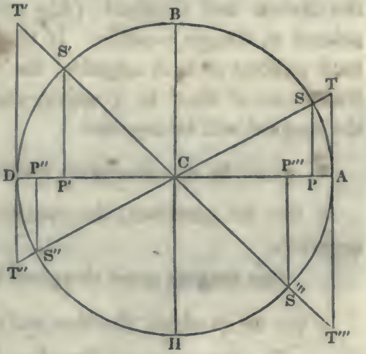
* See note C.

quarters of the circle. The sines, tangents, &c., in the tables, are calculated for a single quadrant only. But these are made to answer for the whole circle. For they are of the same length in each of the four quadrants. (Art. 90.) Some of them however, are *positive;* while others are *negative.* In algebraic processes, this distinction must not be neglected.

193. For the purpose of tracing the changes of the signs, in different parts of the circle, let it be supposed that a straight line CT is fixed at one end C, while the other end is carried round, like a rod moving on a pivot; so that the point S shall describe the circle ABDH. If the two diameters AD and BH, be perpendicular to each other, they will divide the circle into quadrants.



194. In the *first quadrant* AB, the sine, cosine, tangent, &c., are considered *all positive.* In the *second quadrant* BD, the sine P′S′ continues *positive;* because it is still on the *upper* side of the diameter AD, from which it is measured. But the *cosine,* which is measured from BH, becomes *negative,* as soon as it changes from the *right* to the *left* of this line. (Alg. 382.) In the *third quadrant* the *sine* becomes negative, by changing from the upper side to the under side of DA. The *cosine* continues negative, being still on the left of BH. In the *fourth quadrant,* the *sine* continues negative. But the *cosine* becomes positive, by passing to the right of BH.

195. The signs of the *tangents* and *secants* may be derived from those of the sines and cosines. The relations of these several lines to each other must be such, that a uniform method of calculation may extend through the different quadrants.

In the first quadrant, (Art. 93. Propor. 1.)

$$R : \cos :: \tan : \sin, \text{ that is, } \operatorname{Tan} = \frac{R \times \sin}{\cos}$$

The sign of the quotient is determined from the signs of the divisor and dividend. (Alg. 100.) The radius is considered as always positive. If then the sine and cosine be both positive or both negative, the tangent will be positive. But if one of these be positive, while the other is negative, the tangent will be negative.

Now by the preceding article,

In the 2d quadrant, the sine is positive, and the cosine negative.

The tangent must therefore be *negative.*

In the 3d quadrant, the sine and cosine are both negative.

The tangent must therefore be *positive.*

In the 4th quadrant, the sine is negative, and the cosine positive.

The tangent must therefore be *negative.*

196. By the 9th, 3d, and 6th proportions in Art. 93.

1. $\operatorname{Tan} : R :: R : \cot$, that is $\operatorname{Cot} = \frac{R^2}{\tan}$.

Therefore, as radius is uniformly positive, the *cotangent* must have the same sign as the tangent.

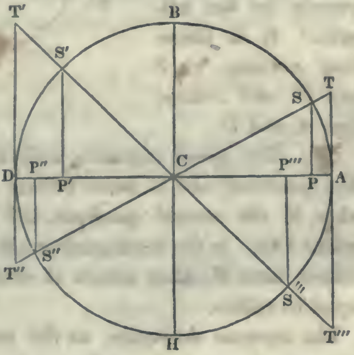
2. $\operatorname{Cos} : R :: R : \sec$, that is, $\operatorname{Sec} = \frac{R^2}{\cos}$.

The *secant*, therefore, must have the same sign as the cosine.

3. Sin : R : : R : cosec, that is, $\text{Cosec}=\frac{R^2}{\sin}$.

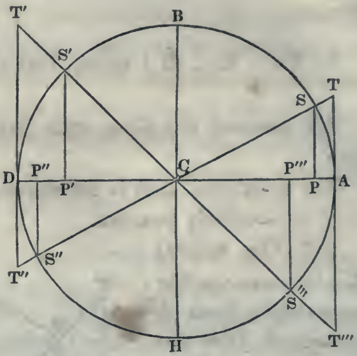
The *cosecant*, therefore, must have the same sign as the sine.

The *versed sine*, as it is measured from A, in one direction only, is invariably positive.



197. The *tangent* AT increases, as the arc extends from A towards B. (See also Fig 11. p. 69.) Near B the increase is very rapid; and when the difference between the arc and 90°, is *less* than any assignable quantity, the tangent is *greater* than any assignable quantity, and is said to be *infinite.* (Alg. 447.) If the arc is *exactly* 90 degrees, it has, strictly speaking, *no* tangent. For a tangent is a line drawn perpendicular to the diameter which passes through one end of the arc, and extended till it *meets* a line proceeding from the centre through the other end. (Art. 84.) But if the arc is 90 degrees, as AB, the angle ACB is a right angle, and therefore AT is *parallel* to CB; so that, if these lines be extended ever so far, they never can meet. Still, as an arc infinitely near to 90° has a tangent infinitely great, it is frequently said, in concise terms, that the tangent of 90° is infinite.

In the second quadrant, the tangent is, at first, infinitely



great, and gradually diminishes, till at D it is reduced to nothing. In the third quadrant, it increases again, becomes infinite near H, and is reduced to nothing at A.

The *cotangent* is inversely as the tangent. It is therefore nothing at B and H, and infinite near A and D.

198. The *secant* increases with the tangent, through the first quadrant, and becomes infinite near B; it then diminishes, in the second quadrant, till at D it is equal to the radius CD. In the third quadrant it increases again, becomes infinite near H, after which it diminishes, till it becomes equal to radius.

The *cosecant* decreases, as the secant increases, and *v. v.* It is therefore equal to radius at B and H, and infinite near A and D.

199. The *sine* increases through the first quadrant, till at B it is equal to radius. (See also Fig. 13. page 70.) It then diminishes, and is reduced to nothing at D. In the third quadrant, it increases again, becomes equal to radius at H, and is reduced to nothing at A.

The *cosine* decreases through the first quadrant, and is reduced to nothing at B. In the second quadrant, it increases, till it becomes equal to radius at D. It then diminishes again, is reduced to nothing at H, and afterwards increases till it becomes equal to radius at A.

In all these cases, the arc is supposed to *begin* at A, and to extend round in the direction of BDH.

200. The *sine* and *cosine* vary from nothing to radius, which they never exceed. The *secant* and *cosecant* are never less than radius, but may be greater than any given length.

The *tangent* and *cotangent* have every value from nothing to infinity. Each of these lines, after reaching its *greatest* limit, begins to *decrease;* and as soon as it arrives at its *least* limit, begins to *increase.* Thus, the sine begins to decrease, after becoming equal to radius, which is its greatest limit. But the secant begins to increase after becoming equal to radius, which is its least limit.

201. The substance of several of the preceding articles is comprised in the following tables. The first shows the *signs* of the trigonometrical lines, in each of the quadrants of the circle. The other gives the *values* of these lines, at the extremity of each quadrant.

Quadrant	1st	2d	3d	4th
Sine and cosecant	+	+	—	—
Cosine and secant	+	—	—	+
Tangent and cotangent	+	—	+	—

	0°	90°	180°	270°	360°
Sine	0	*r*	0	*r*	0
Cosine	*r*	0	*r*	0	*r*
Tangent	0	∝	0	∝	0
Cotangent	∝	0	∝	0	∝
Secant	*r*	∝	*r*	∝	*r*
Cosecant	∝	*r*	∝	*r*	∝

Here *r* is put for radius, and ∝ for infinite.

202. By comparing these two tables, it will be seen, that each of the trigonometrical lines changes from positive to negative, or from negative to positive, in that part of the circle in which the line is either *nothing* or *infinite.* Thus,

the tangent changes from positive to negative, in passing from the first quadrant to the second, through the place where it is infinite. It becomes positive again, in passing from the second quadrant to the third, through the point in which it is nothing.

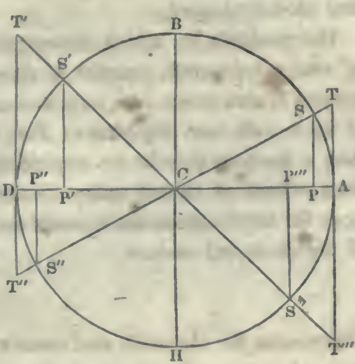
203. There can be no more than 360 degrees in any circle. But a body may have a number of successive revolutions in the same circle; as the earth moves round the sun, nearly in the same orbit, year after year. In astronomical calculations, it is frequently necessary to add together parts of different revolutions. The sum may be more than 360°. But a body which has made more than a complete revolution in a circle, is only brought back to a point which it had passed over before. So the sine, tangent, &c., of an arc greater than 360°, is the same as the sine, tangent, &c., of some arc less than 360°. If an entire circumference, or a number of circumferences, be added to any arc, it will terminate in the same point as before. So that, if C be put for a whole circumference, or 360°, and x be any arc whatever;

$$\sin x = \sin (C+x) = \sin (2C+x) = \sin (3C+x), \&c.$$
$$\tan x = \tan (C+x) = \tan (2C+x) = \tan (3C+x), \&c.$$

204. It is evident also, that, in a number of successive revolutions, in the same circle;

The first quadrant must coincide with the	5th, 9th, 13th, 17th,
The second, with the	6th, 10th, 14th, 18th, &c.
The third, with the	7th, 11th, 15th, 19th, &c.
The fourth, with the	8th, 12th, 16th, 20th, &c.

205. If an arc, extending in a certain direction from a given point, be considered *positive;* an arc extending from the same point, in an *opposite* direction, is to be considered



negative. (Alg. 382.) Thus, if the arc extending from A to S, be positive; an arc extending from A to S''' will be negative. The latter will not terminate in the *same quadrant* as the other —and the signs of the tabular lines must be accommodated to this circumstance. Thus, the sine of AS will be positive, while that of AS''' will be negative. (Art. 194.) When a greater arc is subtracted from a less, if the latter be positive, the *remainder* must be negative. (Alg. 40.)

TRIGONOMETRICAL FORMULÆ.

206. From the view which has been here taken of the changes in the trigonometrical lines, it will be easy to see, in what parts of the circle each of them increases or decreases. But this does not determine their exact values, except at the extremities of the several quadrants. In the analytical investigations which are carried on by means of these lines, it is necessary to calculate the changes produced in them, by a given increase or diminution of the arcs to which they belong. In this there would be no difficulty, if the sines, tangents, &c., were *proportioned* to their arcs. But this is far from being the case. If an arc is doubled, its sine is *not* exactly doubled. Neither is its tangent or secant. We have to inquire, then, in what manner the sine,

tangent, &c., of one arc may be obtained, from those of other arcs already known.

The problem on which almost the whole of this branch of analysis depends, consists in deriving, from the sines and cosines of two given arcs, expressions for the sine and cosine of their *sum* and *difference.* For, by addition and subtraction, a few arcs may be so combined and varied, as to produce others of almost every dimension. And the expressions for the tangents and secants may be deduced from those of the sines and cosines.

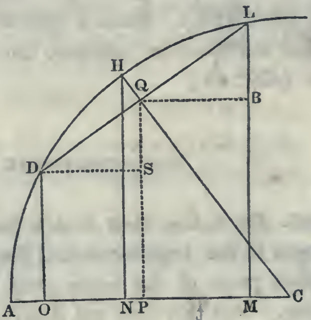
Expressions for the SINE *and* COSINE *of the* SUM *and* DIFFERENCE *of arcs.*

207. Let a=AH, the greater of the given arcs,

And b=HL=HD, the less.

Then $a+b$=AH+HL=AL, the sum of the two arcs,

And $a-b$=AH−HD=AD, their *difference.*



Draw the chord DL, and the radius CH, which may be represented by R. As DH is, by construction, equal to HL; DQ is equal to QL, and therefore DL is perpendicular to CH. (Euc. 3. 3.) Draw DO, HN, QP, and LM, each perpendicular to AC; and DS and QB parallel to AC.

From the definitions of the sine and cosine, (Arts. 82, 9.) it is evident, that

$$\text{The sine}\left\{\begin{array}{lr}\text{of AH, that is,} & \sin a=HN,\\ \text{of HL,} & \sin b=QL,\\ \text{of AL,} & \sin(a+b)=LM,\\ \text{of AD,} & \sin(a-b)=DO,\end{array}\right.$$

$$\text{The cosine}\left\{\begin{array}{lr}\text{of AH, that is,} & \cos a=CN,\\ \text{of HL,} & \cos b=CQ,\\ \text{of AL,} & \cos(a+b)=CM,\\ \text{of AD,} & \cos(a-b)=CO.\end{array}\right.$$

The triangle CHN is obviously similar to CQP; and it is also similar to BLQ, because the sides of the one are perpendicular to those of the other, each to each. We have, then,

1. $CH : CQ :: HN : QP$, that is, $R : \cos b :: \sin a : QP$,
2. $CH : QL :: CN : BL$, $R : \sin b :: \cos a : BL$,
3. $CH : CQ :: CN : CP$ $R : \cos b :: \cos a : CP$,
4. $CH : QL :: HN : QB$, $R : \sin b :: \sin a : QB$,

Converting each of these proportions into an equation;

$$1.\ QP=\frac{\sin a\cos b^{*}}{R} \qquad 3.\ CP=\frac{\cos a\cos b}{R}$$

$$2.\ BL=\frac{\sin b\cos a}{R} \qquad 4.\ QB=\frac{\sin a\sin b}{R}$$

Then adding the first and second,

$$QP+BL=\frac{\sin a\cos b+\sin b\cos a}{R}$$

Subtracting the second from the first,

$$QP-BL=\frac{\sin a\cos b-\sin b\cos a}{R}$$

---

[*] In these formulæ, the sign of multiplication is omitted; $\sin a\cos b$ being put for $\sin a\times\cos b$, that is, the product of the sine of a into the cosine of b.

Subtracting the fourth from the third,

$$CP-QB=\frac{\cos a \cos b-\sin a \sin b}{R}$$

Adding the third and fourth,

$$CP+QB=\frac{\cos a \cos b+\sin a \sin b}{R}$$

But it will be seen, from the figure, that

$$QP+BL=BM+BL=LM=\sin (a+b)$$
$$QP-BL=QP-QS=DO=\sin (a-b)$$
$$CP-QB=CP-PM=CM=\cos (a+b)$$
$$CP+QB=CP+SD=CO=\cos (a-b)$$

208. If then, for the first member of each of the four equations above, we substitute its value, we shall have,

$$\text{I. } \sin (a+b=\frac{\sin a \cos b+\sin b \cos a}{R}$$

$$\text{II. } \sin (a-b)=\frac{\sin a \cos b-\sin b \cos a}{R}$$

$$\text{III. } \cos (a+b)=\frac{\cos a \cos b-\sin a \sin b}{R}$$

$$\text{IV. } \cos (a-b)=\frac{\cos a \cos b+\sin a \sin b}{R}$$

Or multiplying both sides by R,

$$R \sin (a+b)=\sin a \cos b+\sin b \cos a$$
$$R \sin (a-b)=\sin a \cos b-\sin b \cos a$$
$$R \cos (a+b)=\cos a \cos b-\sin a \sin b$$
$$R \cos (a-b)=\cos a \cos b+\sin a \sin b$$

That is, the product of radius and the *sine* of the *sum* of two arcs, is equal to the product of the sine of the first arc into the cosine of the second + the product of the sine of the second into the cosine of the first.

The product of radius and the *sine* of the *difference* of two arcs, is equal to the product of the sine of the first arc into the cosine of the second — the product of the sine of the second into the cosine of the first.

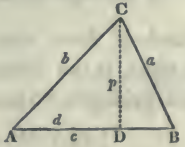
The product of radius and the *cosine* of the *sum* of two arcs, is equal to the product of the cosines of the arcs — the product of their sines.

The product of radius and the *cosine* of the *difference* of two arcs, is equal to the product of the cosines of the arcs + the product of their sines.

These four equations may be considered as fundamental propositions, in what is called the *Arithmetic* of *Sines and Cosines,* or *Trigonometrical Analysis.*

Expression for the *area* of a triangle, in terms of the sides.

209. Let the sides of the triangle ABC be expressed by a, b, and c, the perpendicular CD by p, the segment AD by d, and the area by S.



Then $a^2=b^2+c^2-2cd$, (Euc. 13. 2.)

Transposing and dividing by $2c$;

$$d=\frac{b^2+c^2-a^2}{2c}.\ \text{Therefore}\ d^2=\frac{(b^2+c^2-a^2)^2}{4c^2}.\ \text{(Alg. 171.)}$$

By Euc. 47, 1, $p^2=b^2-d^2=b^2-\frac{(b^2+c^2-a^2)^2}{4c^2}$

Reducing the fraction, (Alg. 120.) and extracting the root of both sides,

$$p=\frac{\sqrt{4b^2c^2-(b^2+c^2-a^2)^2}}{2c}$$ [*]

This gives the length of the *perpendicular* in terms of the sides of the triangle. But the *area* is equal to the product of the base into half the perpendicular height. (Alg. 393.) That is,

$$S=\tfrac{1}{2}cp=\tfrac{1}{4}\sqrt{4b^2c^2-(b^2+c^2-a^2)^2}$$

Here we have an expression for the area, in terms of the sides. But this may be reduced to a form much better adapted to arithmetical computation. It will be seen, that the quantities $4b^2c^2$, and $(b^2+c^2-a^2)^2$ are both *squares;* and that the whole expression under the radical sign is the *difference* of these squares. But the difference of two squares is equal to the product of the sum and difference of their roots. (Alg. 191.) Therefore, $4b^2c^2-(b^2+c^2-a^2)^2$ may be resolved into the two factors,

$$\left\{\begin{array}{l} 2bc+(b^2+c^2-a^2) \text{ which is equal to } (b+c)^2-a^2 \\ 2bc-(b^2+c^2-a^2) \text{ which is equal to } a^2-(b-c)^2 \end{array}\right.$$

Each of these also, as will be seen in the expressions on the right, is the difference of two squares; and may, on the same principle, be resolved into factors, so that,

$$\left\{\begin{array}{l} (b+c)^2-a^2=(b+c+a)\times(b+c-a) \\ a^2-(b-c)^2=(a+b-c)\times(a-b+c) \end{array}\right.$$

---

[*] The expression for the perpendicular is the same, when one of the angles is *obtuse*, as in Fig. 24. page 86. Let $AD=d$.

Then $a^2=b^2+c^2+2cd$. (Euc. 12, 2.) And $d=\frac{-b^2-c^2+a^2}{2c}$

Therefore, $d^2=\frac{(-b^2-c^2+a^2)^2}{4c^2}=\frac{(b^2+c^2-a^2)^2}{4c^2}$ (Alg. 169.)

And $p=\frac{\sqrt{4b^2c^2-(b^2+c^2-a^2)^2}}{2c.}$ as above

Substituting, then, these four factors, in the place of the quantity which has been resolved into them, we have,

$$S=\tfrac{1}{4}\sqrt{(b+c+a)\times(b+c-a)\times(a+b-c)\times(a-b+c,}$$

Here it will be observed, that all the three sides, a, b, and c, are in each of these factors.

Let $h=\frac{1}{2}(a+b+c)$ *half the sum* of the sides. Then

$$S=\sqrt{h\times(h-a)\times(h-b)\times(h-c)}$$

210. For finding the area of a triangle, then, when the three sides are given, we have this general rule;

From half the sum of the sides, subtract each side severally; multiply together the half sum and the three remainders; and extract the square root of the product.

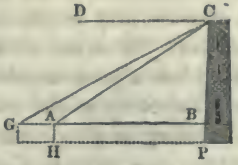
APPLICATION OF TRIGONOMETRY
TO THE
MENSURATION
OF
HEIGHTS AND DISTANCES.

Art. 1. The most direct and obvious method of determining the distance or height of any object, is to apply to it some known measure of length, as a foot, a yard, or a rod. In this manner, the height of a room is found, by a joiner's rule; or the side of a field by a surveyor's chain. But in many instances, the object, or a part, at least, of the line which is to be measured, is *inaccessible.* We may wish to determine the breadth of a river, the height of a cloud, or the distances of the heavenly bodies. In such cases it is necessary to measure some *other* line; from which the required line may be obtained, by geometrical construction, or more exactly, by trigonometrical calculation. The line first measured is frequently called a *base* line.

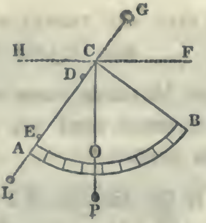
2. In measuring *angles,* some instrument is used which contains a portion of a graduated circle divided into degrees and minutes. For the proper measure of an angle is an arc of a circle, whose centre is the angular point. (Trig. 74.) The instruments used for this purpose are made in different forms, and with various appendages. The essential parts are a graduated circle, and an index with sight-holes, for taking the directions of the lines which include the angles.

3. Angles of *elevation,* and of *depression* are in a plane

perpendicular to the horizon, which is called a *vertical plane.* An angle of *elevation* is contained between a parallel to the horizon, and an ascending line, as BAC. An angle of *depression* is contained between a parallel to the horizon, and a descending line, as DCA. The *complement* of this is the angle ACB.



4. The instrument by which angles of elevation, and of depression, are commonly measured, is called a *Quadrant.* In its most simple form, it is a portion of a circular board ABC, on which is a graduated arc of 90 degrees, AB, a plumb line CP, suspended from the central point C, and two sight-holes D and E, for taking the direction of the object.



To measure an angle of *elevation* with this, hold the plane of the instrument perpendicular to the horizon, bring the centre C to the angular point, and direct the edge AC in such a manner, that the object G may be seen through the two sight-holes. Then the arc BO measures the angle BCO, which is equal to the angle of elevation FCG. For as the plumb line is perpendicular to the horizon, the angle FCO is a right angle, and therefore equal to BCG. Taking from these the common angle BCF, there will remain the angle BCO=FCG.

In taking an angle of *depression,* as HCL, the eye is placed at C, so as to view the object at L, through the sight-holes D and E.

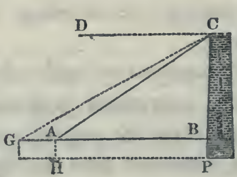
5. In treating of the mensuration of heights and dis-

tances, no *new principles* are to be brought into view. We have only to make an application of the rules for the solution of triangles, to the particular circumstances in which the observer may be placed, with respect to the line to be measured. These are so numerous, that the subject may be divided into a great number of distinct cases. But as they are all solved upon the same general principles, it will not be necessary to give examples under each. The following problems may serve as a specimen of those which most frequently occur in practice.

PROBLEM I.

TO FIND THE PERPENDICULAR HEIGHT OF AN ACCESSIBLE OBJECT STANDING ON A HORIZONTAL PLANE.

6. MEASURE FROM THE OBJECT TO A CONVENIENT STATION, AND THERE TAKE THE ANGLE OF ELEVATION SUBTENDED BY THE OBJECT.



If the distance AB be measured, and the angle of elevation BAC; there will be given in the right angled triangle ABC, the base and the angles, to find the perpendicular. (Trig. 137.)

As the instrument by which the angle at A is measured, is commonly raised a few feet above the ground; a point B must be taken in the object, so that AB shall be parallel to the horizon. The part BP, may afterwards be added to the height BC, found by trigonometrical calculation.

Ex. 1. What is the height of a tower BC, if the distance AB, on a horizontal plane, be 98 feet; and the angle BAC $35\frac{1}{2}$ degrees?

Making the hypothenuse radius, (Trig. 121.)

Cos. BAC : AB : : Sin. BAC : BC=69.9 feet.

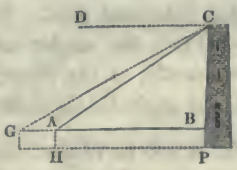
For the *geometrical construction* of the problem, see Trig. 169.

2. What is the height of the perpendicular sheet of water at the falls of Niagara, if it subtends an angle of 40 degrees, at the distance of 163 feet from the bottom, measured on a horizontal plane? Ans. 136$\frac{3}{4}$ feet.

7. If the height of the object be *known*, its *distance* may be found by the angle of elevation. In this case the angles, and the perpendicular of the triangle are given, to find the base.

Ex. A person on shore, taking an observation of a ship's mast, which is known to be 99 feet high, finds the angle of elevation 3$\frac{1}{2}$ degrees. What is the distance of the ship from the observer? Ans. 98 rods.

8. If the observer be stationed at the *top* of the perpendicular BC, whose height is known; he may find the length of the base line AB, by measuring the angle of *depression* ACD, which is equal to BAC.



Ex. A seaman at the top of a mast 66 feet high, looking at another ship, finds the angle of depression 10 degrees. What is the distance of the two vessels from each other?

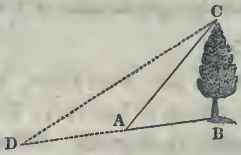
Ans. 22$\frac{2}{3}$ rods.

We may find the distance between *two objects* which are in the same vertical plane with the perpendicular, by calculating the distance of each from the perpendicular. Thus AG is equal to the difference between AB and GB.

PROBLEM II.

TO FIND THE HEIGHT OF AN ACCESSIBLE OBJECT STANDING ON AN INCLINED PLANE.

9. MEASURE THE DISTANCE FROM THE OBJECT TO A CONVENIENT STATION, AND TAKE THE ANGLES WHICH THIS BASE MAKES WITH LINES DRAWN FROM ITS TWO ENDS TO THE TOP OF THE OBJECT.



If the base AB be measured and the angles BAC and ABC; there will be given, in the oblique angled triangle ABC, the side AB, and the angles, to find BC. (Trig. 150.)

Or the height BC may be found by measuring the distances BA, AD, and taking the angles, BAC and BDC. There will then be given in the triangle ADC, the angles and the side AD, to find AC; and consequently, in the triangle ABC, the sides AB and AC with the angle BAC, to find BC.

Ex. If AB be 76 feet, the angle B 101° 25′, and the angle A 44° 42′; what is the height of the tree BC?

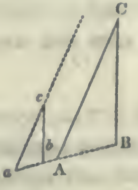
Sin. C : AB : : Sin. A : : BC=95.9 feet.

For the *geometrical construction* of the problem, see Trig. 169.

10. The following are some of the methods by which the height of an object may be found, without measuring the angle of elevation.

1. *By shadows.* Let the staff *bc* be parallel to an ob-

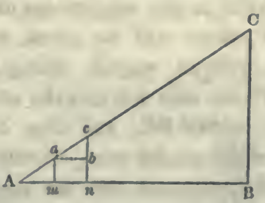
ject BC, whose height is required. If the shadow of BC extend to A, and that of *bc* to *a*; the rays of light CA and *ca* coming from the sun may be considered parallel; and therefore the triangles ABC and *abc* are similar; so that



$$ab : bc :: AB : BC.$$

Ex. If *ab* be 3 feet, *bc* 5 feet, and AB 69 feet, what is the height of BC? Ans. 115 feet.

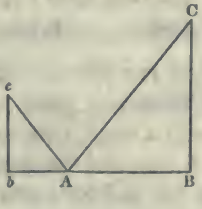
2. *By parallel rods.* If two poles *am* and *cn* be placed parallel to the object BC, and at such distances as to bring the points C, *c*, *a*, in a line, and if *ab* be made parallel to AB; the triangles ABC, and *abc* will be similar; and we shall have



$$ab : bc :: AB : BC.$$

One pole will be sufficient, if the observer can place his eye at the point A, so as to bring A, *a*, and C in a line.

3. *By a mirror.* Let the smooth surface of a body of water at A, or any plane mirror parallel to the horizon, be so situated, that the eye of the observer at *c* may view the top of the object C reflected from the mirror. By a law of Optics, the angle BAC is equal to *b*A*c*; and if *bc* be made parallel to BC, the triangle *b*A*c* will be similar to BAC; so that



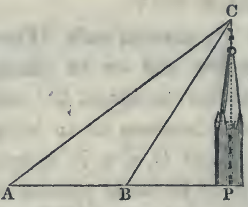
$$Ab : bc :: AB : BC.$$

Problem III.

TO FIND THE HEIGHT OF AN INACCESSIBLE OBJECT ABOVE A HORIZONTAL PLANE.

11. TAKE TWO STATIONS IN A VERTICAL PLANE PASSING THROUGH THE TOP OF THE OBJECT, MEASURE THE DISTANCE FROM ONE STATION TO THE OTHER, AND THE ANGLE OF ELEVATION AT EACH.

If the base AB be measured, with the angle CBP and CAB; as ABC is the supplement of CBP, there will be given, in the oblique angled triangle ABC, the side AB and the angles, to find BC; and then in the right angled triangle BCP, the hypothenuse and the angles, to find the perpendicular CP.



Ex. 1. If C be the top of a spire, the horizontal base line AB 100 feet, the angle of elevation BAC 40°, and the angle PBC 60°; what is the perpendicular height of the spire?

The difference between the angles PBC and BAC is equal to ACB. (Euc. 32. 1.)

Then Sin ACB : AB : : Sin BAC : BC=187.9
And R : BC : : Sin PBC : CP=$162\frac{3}{4}$ feet.

2. If two persons 120 rods from each other, are standing on a horizontal plane, and also in a vertical plane passing through a *cloud*, both being on the same side of the cloud: and if they find the angles of elevation at the two stations to be 68° and 76°; what is the height of the cloud?

Ans. 2 miles 135.7 rods.

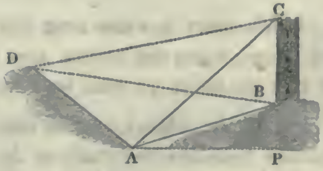
12. The preceding problems are useful in particular cases.

But the following is a *general* rule, which may be used for finding the height of any object whatever, within moderate distances.

Problem IV.

TO FIND THE HEIGHT OF ANY OBJECT, BY OBSERVATIONS AT TWO STATIONS.

13. MEASURE THE BASE LINE BETWEEN THE TWO STATIONS, THE ANGLES BETWEEN THIS BASE AND LINES DRAWN FROM EACH OF THE STATIONS TO EACH END OF THE OBJECT, AND THE ANGLE SUBTENDED BY THE OBJECT, AT ONE OF THE STATIONS.



If BC be the object whose height is required, and if the distance between the stations A and D be measured, with the angles ADC, DAC, ADB, DAB, and BAC; there will be given in the triangle ADC, the side AD and the angles, to find AC; in the triangle ADB, the side AD and the angles, to find AB; and then, in the triangle BAC, the sides AB and AC with the included angle, to find the required height BC.

If the two stations A and D be in the *same plane* with BC, the angle BAC will be equal to the difference between BAD and CAD. In this case it will not be necessary to measure BAC.

$$\text{Ex. If } AD=83 \text{ feet,} \quad \left\{\begin{array}{l} ADC=51^\circ \\ DAC=95^\circ \end{array}\right. \qquad \left\{\begin{array}{l} ADB=33^\circ \\ DAB=121^\circ \end{array}\right. \quad BAC=26^\circ,$$

What is the height of the object BC?

$$\text{Sin } ACD : AD :: ADC : AC = 115.3 \text{ (Fig. 8.)}$$

$$\text{Sin } ABD : AD :: ADB : AB = 103.1.$$

$$(AC+AB) : (AC-AB) :: \text{Tan } \tfrac{1}{2}\,(ABC+ACB) : \text{Tan } \tfrac{1}{2}\,(ABC-ACB) = 13^\circ\, 38'$$

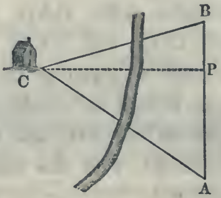
$$\text{Sin } ACB : AB :: \text{Sin } BAC : BC = 50.57 \text{ feet.}$$

If the object BC be perpendicular to the horizon, its height, after obtaining AB and AC as before, may be found by taking the *angles* of *elevation* BAP and CAP. The difference of the perpendiculars in the right angled triangles ABP and ACP, will be the height required.

Problem V.

TO FIND THE DISTANCE OF AN INACCESSIBLE OBJECT.

14. MEASURE A BASE LINE BETWEEN TWO STATIONS, AND THE ANGLES BETWEEN THIS AND LINES DRAWN FROM EACH OF THE STATIONS TO THE OBJECT.



If C be the object, and if the distance between the stations A and B be measured, with the angles at B and A; there will be given, in the oblique angled triangle ABC, the side AB and the angles, to find AC and BC, the distances of the object from the two stations.

For the geometrical construction, see Trig. 169.

Ex. 1. What are the distances of the two stations A and B from the house C, on the opposite side of a river; if AB be 26.6 rods, B 92° 46′, and A 38° 40′?

The angle $C = 180 - (A+B) = 48^\circ\, 34'$. Then

$$\text{Sin } C : AB :: \left\{ \begin{array}{l} \text{Sin } A : BC = 22.17 \\ \text{Sin } B : AC = 35.44. \end{array} \right.$$

2. Two ships in a harbor, wishing to ascertain how far they are from a fort on shore, find that their mutual distance is 90 rods, and that the angles formed between a line from one to the other, and lines drawn from each to the fort are 45° and 56° 15′. What are their respective distances from the fort? Ans. 76.3 and 64.9 rods.

15. The *perpendicular* distance of the object from the line joining the two stations may be easily found, after the distance from one of the stations is obtained. The perpendicular distance PC is one of the sides of the right angled triangle BCP. Therefore

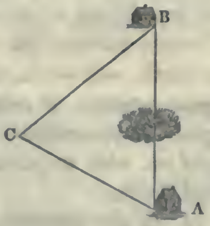
$$R : BC :: \text{Sin } B : PC.$$

PROBLEM VI.

TO FIND THE DISTANCE BETWEEN TWO OBJECTS, WHEN THE PASSAGE FROM ONE TO THE OTHER, IN A STRAIGHT LINE IS OBSTRUCTED.

16. MEASURE THE RIGHT LINES FROM ONE STATION TO EACH OF THE OBJECTS, AND THE ANGLE INCLUDED BETWEEN THESE LINES.

If A and B be the two objects, and if the distances BC and AC be measured, with the angle at C; there will be given, in the oblique angled triangle ABC, two sides and the included angle, to find the other two angles, and the remaining side. (Trig. 153.)



Ex. The passage between the two objects A and B being obstructed by a morass, the line BC was measured and found to be 109 rods, the line AC 76 rods, and the angle at C 101° 30′. What is the distance AB?

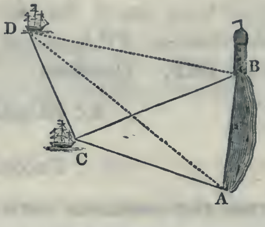
Ans. 144.7 rods.

PROBLEM VII.

TO FIND THE DISTANCE BETWEEN TWO INACCESSIBLE OBJECTS.

17. MEASURE A BASE LINE BETWEEN TWO STATIONS AND THE ANGLES BETWEEN THIS BASE AND LINES DRAWN FROM EACH OF THE STATIONS TO EACH OF THE OBJECTS.

If A and B be the two objects, and if the distance between the stations C and D be measured, with the angles BDC, BCD, ADC, and ACD; the lines AC and BC may be found as in Problem V, and then the distance AB as in Problem VI.

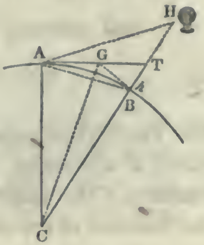


This rule is substantially the same as that in Art. 13. The two stations are supposed to be in the *same plane* with the objects. If they are not, it will be necessary to measure the angle ACB.

18. The same process by which we obtain the distance of *two* objects from each other, will enable us to find the distance between one of these and a third, between that and a fourth, and so on, till a connection is formed between a great number of remote points. This is the plan of the great *Trigonometrical Surveys*, which have been lately carried on, with surprising exactness, particularly in England and France.

19. In the preceding problems for determining altitudes, the objects are supposed to be at such moderate distances, that the observations are not sensibly affected by the *spherical figure of the earth.* The height of an object is measured from an *horizontal plane,* passing through the station at which the angle of elevation is taken. But in an extent

of several miles, the figure of the earth ought to be taken into account.



Let AB be a portion of the earth's surface, H an object above it, and AT a tangent at the point A, or a horizontal line passing through A. Then HT, the oblique height of the object above the horizon of A, is only a *part* of the height above the surface of the earth, or the level of the ocean. To obtain the true altitude, it is necessary to add BT to the height HT found by observation. The height BT may be calculated, if the diameter of the earth and the distance AT be previously known. Or if the height BT be first determined from observation, with the distance AT; the diameter of the earth may be thence deduced.

PROBLEM VIII.

TO FIND THE DIAMETER OF THE EARTH, FROM THE KNOWN HEIGHT OF A DISTANT MOUNTAIN, WHOSE SUMMIT IS JUST VISIBLE IN THE HORIZON.

20. **FROM THE SQUARE OF THE DISTANCE DIVIDED BY THE HEIGHT, SUBTRACT THE HEIGHT.**

If BT (above figure) be a mountain whose height is known, with the distance AT; and if the summit T be just visible in the horizon at A; then AT is a *tangent* at the point A.

Let $2BC=D$, the diameter of the earth,

$AT=d$, the distance of the mountain,

$BT=h$, its height.

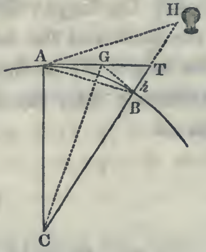
Then considering AT as a straight line, and the earth as a sphere, we have (Euc. 36. 3.)*

$$(2BC+BT)\times BT=AT^2;\ \text{that is, } (D+h)\times h=d^2,$$

and reducing the equation,

$$D=\frac{d^2}{h}-h.$$

Ex. The highest point of the Andes is about 4 miles above the level of the ocean. If a straight line from this touch the surface of the water at the distance of 178¼ miles; what is the diameter of the earth? Ans. 7940 miles.



21. If the distance AT be *unknown*, it may be found by measuring with a quadrant the angle ATC. Draw BG perpendicular to BC; and join CG. The triangles ACG and BCG are equal, because each has a right angle, the sides AC and BC are equal, and the hypothenuse CG is common. Therefore BG and AG are equal. In the right angled triangle BGT, the angle BTG is given, and the perpendicular BT. From these may be found BG and TG, whose sum is equal to AT, the distance required.†

22. In the common measurement of angles, the light is supposed to come from the object to the eye in a *straight line*. But this is not strictly true. The direction of the light is affected by the *refraction of the atmosphere*. If the object be near, the deviation is very inconsiderable. But in an extent of several miles, and particularly in such nice ob-

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* Thomson's Legendre, 30. 4.

† This method of determining the diameter of the earth is not as accurate as that by measuring a degree of Latitude.

servations as determining the height of distant mountains, and the diameter of the earth, it is necessary to make allowance for the refraction.

PROBLEM IX.

TO FIND THE GREATEST DISTANCE AT WHICH A GIVEN OBJECT CAN BE SEEN ON THE SURFACE OF THE EARTH.

23. TO THE PRODUCT OF THE HEIGHT OF THE OBJECT INTO THE DIAMETER OF THE EARTH, ADD THE SQUARE OF THE HEIGHT; AND EXTRACT THE SQUARE ROOT OF THE SUM.

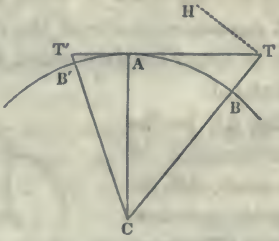
Let $2BC=D$, the diameter of the earth, (Fig. 12.)
$BT=h$, the height of the object,
$AT=d$, the distance required.

$$\text{Then } (D+h)\times h=d^2. \quad \text{And } d=\sqrt{Dh+h^2}.$$

Ex. If the diameter of the earth be 7940 miles, and Mount Ætna 2 miles high; how far can its summit be seen at sea? Ans. 126 miles.

The actual distance at which an object can be seen, is increased by the refraction of the air.

24. In this problem the eye is supposed to be placed at the level of the ocean. But if the observer be elevated above the surface, as on the deck of a ship, he can see to a greater distance. If BT be the height of the object, and B′T′ the height of the eye above the level of the ocean; the distance at which the object can be seen, is evidently equal to the *sum* of the tangents AT and AT′.



Ex. The top of a ship's mast 132 feet high is just visible in the horizon, to an observer whose eye is 33 feet above the surface of the water. What is the distance of the ship?

Ans. $21\frac{1}{8}$ miles.

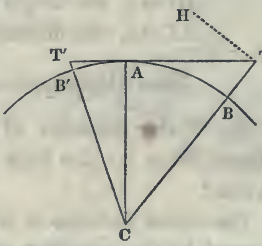
25. The distance to which a person can see the smooth surface of the ocean, if no allowance be made for refraction, is equal to a tangent to the earth drawn from his eye, as T′A. (Fig. 13.)

Ex. If a man standing on the level of the ocean, has his eye raised $5\frac{1}{2}$ feet above the water: to what distance can he see the surface?

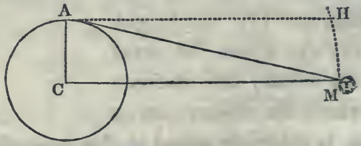
Ans. $2\frac{7}{8}$ miles.

26. If the distance AT, with the diameter of the earth be given, and the *height* BT be required; the equation in Art. 23 gives

$$h=\sqrt{\tfrac{1}{4}D^2+d^2}-\tfrac{1}{2}D$$



27. When the diameter of the earth is ascertained, this may be made a *base line* for determining the distance of the *heavenly bodies.* A right angled triangle may be formed, the perpendicular sides of which shall be the distance required, and the semi-diameter of the earth. If then one of the angles be found by observation, the required side may be easily calculated.



Let AC be the semi-diameter of the earth, AH the sensible horizon at A, and CM the rational horizon, parallel to AH, passing through the moon M. The angle HAM may be found by astronomical observation. This angle, which is called the *Horizontal Parallax,* is equal to AMC, the angle at the moon subtended by the semi-diameter of the earth. (Euc. 29. 1.)

PROBLEM X.

TO FIND THE DISTANCE OF ANY HEAVENLY BODY WHOSE HORIZONTAL PARALLAX IS KNOWN.

28. AS RADIUS, TO THE SEMI-DIAMETER OF THE EARTH; SO IS THE CO-TANGENT OF THE HORIZONTAL PARALLAX, TO THE DISTANCE.

In the right angled triangle ACM, (Fig. 14.) if AC be made radius;

$$R : AC :: \text{Cot. } AMC : CM.$$

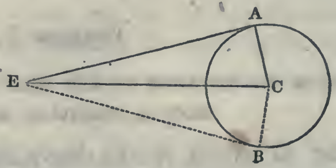
Ex. If the horizontal parallax of the moon be 0° 57′, and the diameter of the earth 7940 miles; what is the distance of the moon from the centre of the earth?

Ans. 239,414 miles.

29. The *fixed stars* are too far distant to have any sensible horizontal parallax. But from late observations it would seem, that some of them are near enough, to suffer a small apparent change of place, from the revolution of the earth round the sun. The distance of the sun, then, which is the semi-diameter of the earth's orbit, may be taken as a *base line,* for finding the distance of the stars.

We thus proceed by degrees from measuring a line on the surface of the earth, to calculate the distances of the heavenly bodies. From a base line on a plane, is determined the height of a mountain; from the height of a mountain, the diameter of the earth; from the diameter of the earth, the distance of the sun, and from the distance of the sun the distance of the stars.

30. After finding the distance of a heavenly body, its *magnitude* is easily ascertained; if it have an apparent diameter, sufficiently large to be measured by the instruments which are used for taking angles.



Let AEB be the angle which a heavenly body subtends at the eye. Half this angle, if C be the centre of the body, is AEC; the line EA is a tangent to the surface, and therefore EAC is a right angle. Then making the distance EC radius,

R : EC : : Sin. AEC : AC.

That is, radius is to the distance, as the sine of half the angle which the body subtends, to its semi-diameter.

Ex. If the sun subtends an angle of 32′ 2″, and if his distance from the earth be 95 million miles; what is his diameter? Ans. 885 thousand miles.

Promiscuous Examples.

1. On the bank of a river, the angle of elevation of a tree on the opposite side is found to be 46°; and at another station 100 feet directly back on the same level, 31°. What is the height of the tree? Ans. 143 feet.

2. On a horizontal plane, observations were taken of a tower standing on the top of a hill. At one station the angle of elevation of the top of the tower was found to be 50°; that at the bottom 39°; and at another station 150 feet directly back, the angle of elevation of the top of the tower was 32°. What are the heights of the hill and the tower?
Ans. The hill is 134 feet high; the tower 63.

3. What is the altitude of the sun, when the shadow of a tree, cast on a horizontal plane, is to the height of the tree as 4 to 3? Ans. 36° 52′ 12″.

4. If a straight line from the top of the White Mountains in New Hampshire touch the ocean at the distance of $103\frac{1}{3}$ miles? what is the height of the mountains?

Ans. 7100 feet.

5. From the top of a perpendicular rock 55 yards high, the angle of depression of the nearest bank of a river is found to be 55° 54′, that of the opposite bank 33° 20′. Required the breadth of the river, and the distance of its nearest bank from the bottom of the rock.

The breadth of the river is 46.4 yards;
Its distance from the rock 37.2.

6. If the moon subtend an angle of 31′ 14″, when her distance is 240,000 miles; what is her diameter?

Ans. 2180 miles.

7. Observations are made on the altitude of a balloon, by two persons standing on the same side of the balloon, and in a vertical plane passing through it. The distance of the stations is half a mile. At one, the angle of elevation is 30° 58′, at the other 36° 52′. What is the height of the balloon above the ground? Ans. $1\frac{1}{2}$ miles.

8. The shadow of the top of a mountain, when the altitude of the sun on the meridian is 32°, strikes a certain point on a level plain below; but when the meridian altitude of the sun is 67°, the shadow strikes half a mile farther south, on the same plain. What is the height of the mountain above the plain? Ans. 2245 feet.

NOTES.

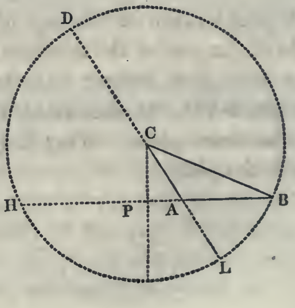
Note A. p. 13.

It is common to *define* logarithms to be a series of numbers in arithmetical progression, corresponding with another series in geometrical progression. This is calculated to perplex the learner, when, upon opening the tables, he finds that the natural numbers, as they stand there, instead of being in *geometrical*, are in *arithmetical* progression; and that the logarithms are *not* in arithmetical progression.

It is true, that a geometrical series may be obtained, by taking out, here and there, a few of the natural numbers; and that the logarithms of these will form an arithmetical series. But the definition is not applicable to the whole of the numbers and logarithms, as they stand in the tables.

Note B. p. 89.

If the perpendicular be drawn from the angle opposite the longest side, it will always fall *within* the triangle; because the other two angles must, of course, be acute. But if one of the angles at the base be *obtuse*, the perpendicular will fall *without* the triangle, as CP.



In this case, the side on which the perpendicular falls, is to the sum of the other two; as the difference of the latter, to the *sum* of the segments made by the perpendicular.

The demonstration is the same, as in the other case, except that AH=BP+PA, instead of BP—PA.

Thus, in the circle BDHL, of which C is the centre,

$$AB\times AH=AL\times AD\text{; therefore } AB : AD :: AL :: AH.$$

$$\begin{aligned}\text{But } AD&=CD+CA=CB+CA\\ \text{And } AL&=CL-CA=CB-CA\\ \text{And } AH&=HP+PA=BP+PA\end{aligned}$$

Therefore,

$$AB : CB+CA :: CB-CA : BP+PA$$

When the three sides are given, it may be known whether one of the angles is obtuse. For any angle of a triangle is obtuse or acute, according as the square of the side subtending the angle is *greater*, or *less*, than the sum of the squares of the sides containing the angle. (Euc. 12, 13. 2.)*

NOTE C. p. 000.

Gunter's *Sliding Rule*, is constructed upon the same principle as his scale, with the addition of a slider, which is so contrived as to answer the purpose of a pair of dividers, in working proportions, multiplying, dividing, &c. The lines on the *fixed part* are the same as on the scale. The *slider* contains two lines of numbers, a line of logarithmic sines, and a line of logarithmic tangents.

To *multiply* by this, bring 1 on the slider, against one of the factors on the fixed part; and against the other factor on the slider, will be the product on the fixed part. To divide, bring the divisor on the slider, against the dividend on the fixed part; and against 1 on the slider, will be the quotient on the fixed part. To work a *proportion*, bring the first term on the slider, against one of the middle terms on the fixed part; and against the other middle term on the slider, will be the fourth term on the fixed part. Or the first term may be taken on the fixed part; and then the fourth term will be found on the slider.

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* Thomson's Legendre, 12, 13. 4.

Another instrument frequently used in trigonometrical constructions, is

THE SECTOR.

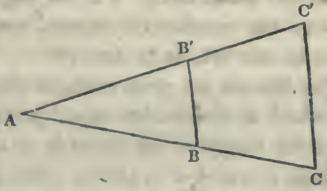
This consists of two equal scales movable about a point as a centre. The lines which are drawn on it are of two kinds, some being parallel to the sides of the instrument, and others diverging from the central point, like the radii of a circle. The latter are called the *double* lines, as each is repeated upon the two scales. The *single* lines are of the same nature, and have the same use, as those which are put upon the common scale; as the lines of equal parts, of chords, of latitude, &c., on one face; and the logarithmic lines of numbers, of sines, and of tangents, on the other.

The *double* lines are

A line of *Lines*, or equal parts, marked	Lin. or L.
A line of *Chords*,	Cho. or C.
A line of natural *Sines*,	Sin. or S.
A line of natural *Tangents* to 45°,	Tan. or T.
A line of tangents, *above* 45°,	Tan. or T.
A line of natural *Secants*,	Sec. or S.
A line of *Polygons*,	Pol. or P.

The double lines of *chords*, of *sines*, and of *tangents* to 45°, are all of the same radius; beginning at the central point, and terminating near the other extremity of each scale; the chords at 60°, the sines at 90°, and the tangents at 45°. (See Art. 95.) The line of *lines* is also of the same length, containing ten equal parts which are numbered, and which are again subdivided. The radius of the lines of secants and of tangents above 45°, is about one-fourth of the length of the other lines. From the end of the radius, which for the secants is at 0, and for the tangents at 45°, these lines extend to between 70° and 80°. The line of polygons is numbered 4, 5, 6, &c., from the extremity of each scale, towards the centre.

The simple principle on which the utility of these several pairs of lines depends is this, that *the sides of similar triangles are proportional.* (Euc. 4. 6.) So that sines, tangents, &c., are furnished *to any radius,* within the extent of the opening of the two scales. Let AC and AC′ be any pair of lines on the sector, and AB and AB′ equal portions of these lines. As AC and AC′ are equal, the triangle ACC′ is isosceles, and similar to ABB′. Therefore,



$$AB : AC :: BB' : CC'.$$

Distances measured from the centre on either scale, as AB and AC, are called *lateral distances.* And the distances between corresponding points of the two scales, as BB′ and CC′, are called *transverse distances.*

Let AC and CC′ be radii of two circles. Then if AB be the chord, sine, tangent, or secant, of any number of degrees in one; BB′ will be the chord, sine, tangent, or secant of the same number of degrees in the other. (Art. 119.) Thus, to find the *chord* of 30°, to a radius of four inches, open the sector so as to make the transverse distance from 60 to 60, on the lines of chords, four inches; and the distance from 30 to 30, on the same lines, will be the chord required. To find the *sine* of 28°, make the distance from 90 to 90, on the lines of sines, equal to radius; and the distance from 28 to 28 will be the sine. To find the *tangent* of 37°, make the distance from 45 to 45, on the lines of tangents, equal to radius; and the distance from 37 to 37 will be the tangent. In finding *secants*, the distance from 0 to 0 must be made radius. (Art. 201.)

To lay down an *angle* of 34°, describe a circle, of any convenient radius, open the sector, so that the distance from 60 to 60 on the lines of chords shall be equal to this radius, and to the circle apply a chord equal to the distance from 34 to 34. (Art. 161.) For an angle above 60°, the chord of *half* the number of degrees may be taken, and applied *twice* on the arc, as in Art. 161.

The line of *polygons* contains the chords of arcs of a circle which is divided into equal portions. Thus, the distances from the centre of the sector to 4, 5, 6, and 7, are the chords of $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, and $\frac{1}{7}$ of a circle. The distance 6 is the radius. (Art. 95.) This line is used to make a regular polygon, or to inscribe one in a given circle. Thus, to make a *pentagon* with the transverse distance from 6 to 6 for radius, describe a circle, and the distance from 5 to 5 will be the length of one of the sides of a pentagon inscribed in that circle.

The line of *lines* is used to divide a line into equal or proportional parts, to find fourth proportionals, &c. Thus, to divide a line into 7 equal parts, make the length of the given line the transverse distance from 7 to 7, and the distance from 1 to 1 will be one of the parts. To find $\frac{3}{5}$ of a line, make the transverse distance from 5 to 5 equal to the given line; and the distance from 3 to 3 will be $\frac{3}{5}$ of it.

In working the *proportions in trigonometry* on the sector, the lengths of the sides of triangles are taken from the line of lines, and the degrees and minutes from the lines of sines, tangents, or secants. Thus, in Art. 135, ex. 1,

$$35 : R :: 26 : \sin 48°.$$

To find the fourth term of this proportion by the sector, make the lateral distance 35 on the line of lines, a transverse distance from 90 to 90 on the lines of sines; then the lateral distance 26 on the line of lines, will be the transverse distance from 48 to 48 on the lines of sines.

For a more particular account of the construction and uses of the Sector, see Stone's edition of Bion on Mathematical Instruments, Hutton's Dictionary, and Robertson's Treatise on Mathematical Instruments.

	44 Degrees.				44 Degrees.		
M	N. Tan.	N. Cot.	M	M	N. Tan.	N. Cot.	M
0	96569	1.03553	60	31	98327	1.01702	29
1	96625	1.03493	59	32	98384	1.01642	28
2	96681	1.03433	58	33	98441	1.01583	27
3	96738	1.03372	57	34	98499	1.01524	26
4	96794	1.03312	56	35	98556	1.01465	25
5	96850	1.03252	55	36	98613	1.01406	24
6	96907	1.03192	54	37	98671	1.01347	23
7	96963	1.03132	53	38	98728	1.01288	22
8	97020	1.03072	52	39	98786	1.01229	21
9	97076	1.03012	51	40	98843	1.01170	20
10	97133	1.02952	50	41	98901	1.01112	19
11	97189	1.02892	49	42	98958	1.01053	18
12	97246	1.02832	48	43	99016	1.00994	17
13	97302	1.02772	47	44	99073	1.00935	16
14	97359	1.02713	46	45	99131	1.00876	15
15	97416	1.02653	45	46	99189	1.00818	14
16	97472	1.02593	44	47	99247	1.00759	13
17	97529	1.02533	43	48	99304	1.00701	12
18	97586	1.02474	42	49	99362	1.00642	11
19	97643	1.02414	41	50	99420	1.00583	10
20	97700	1.02355	40	51	99478	1.00525	9
21	97756	1.02295	39	52	99536	1.00467	8
22	97813	1.02236	38	53	99594	1.00408	7
23	97870	1.02176	37	54	99652	1.00350	6
24	97927	1.02117	36	55	99710	1.00291	5
25	97984	1.02057	35	56	99768	1.00233	4
26	98041	1.01998	34	57	99826	1.00175	3
27	98098	1.01939	33	58	99884	1.00116	2
28	98155	1.01879	32	59	99942	1.00058	1
29	98213	1.01820	31	60	10000	1.00000	0
30	98270	1.01761	30				
M	N. Cot.	N. Tan.	M	M	N. Cot.	N. Tan.	M
	45 Degrees.				45 Degrees.		

38

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